MDS codes with optimal regeneration

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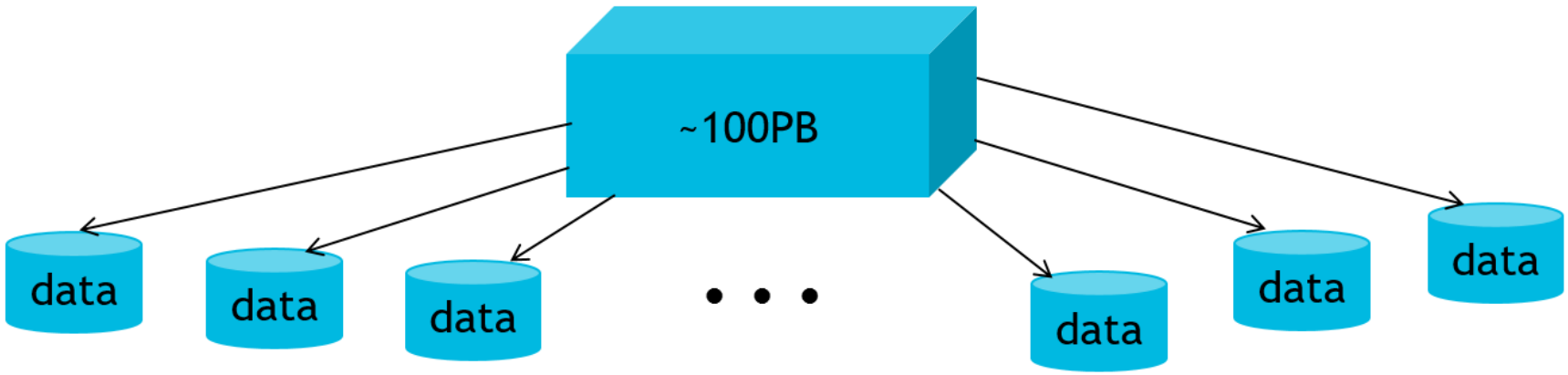
June 21, 2017

Motivation: Distributed Storage Systems (DSS)



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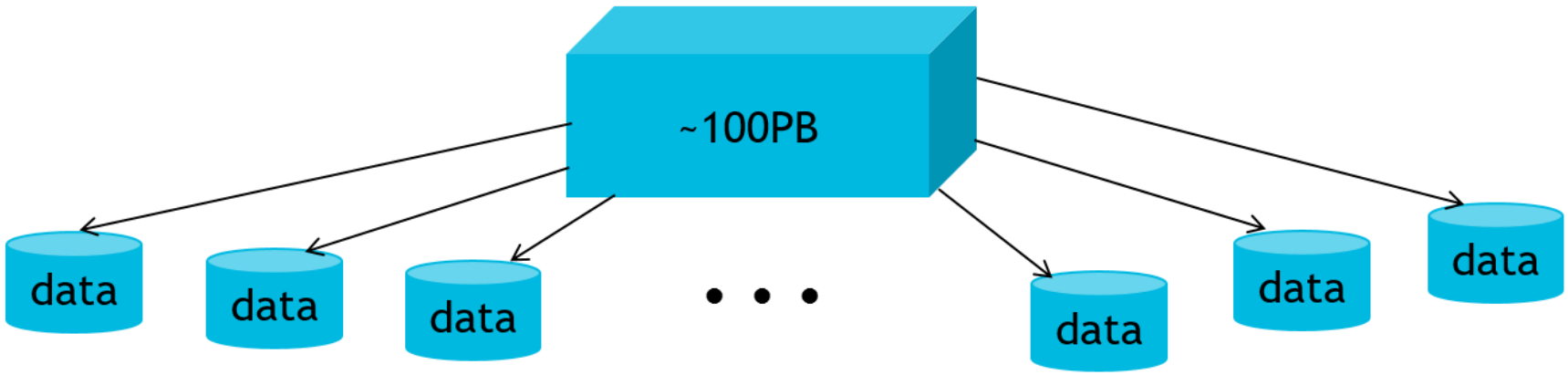

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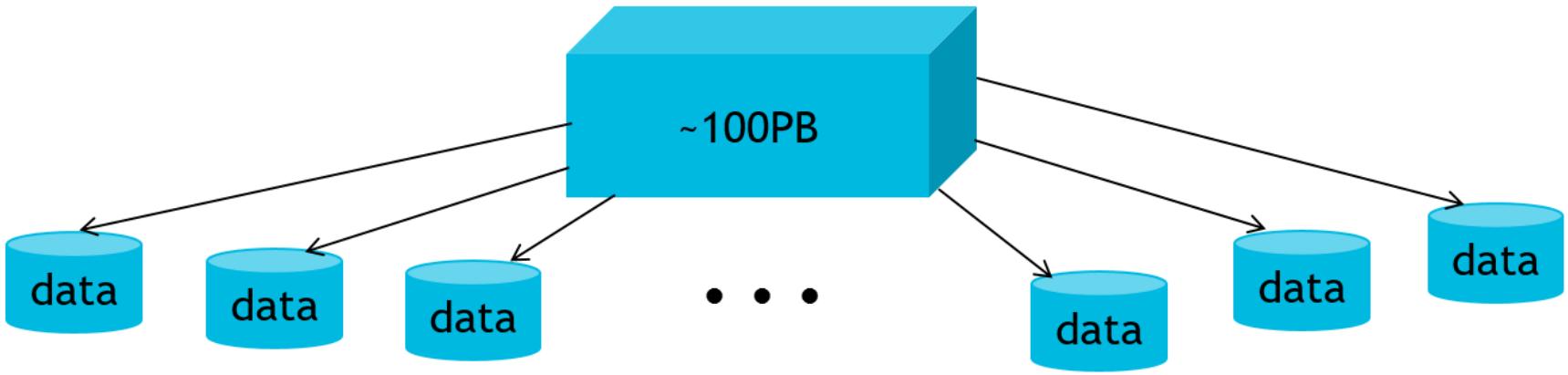
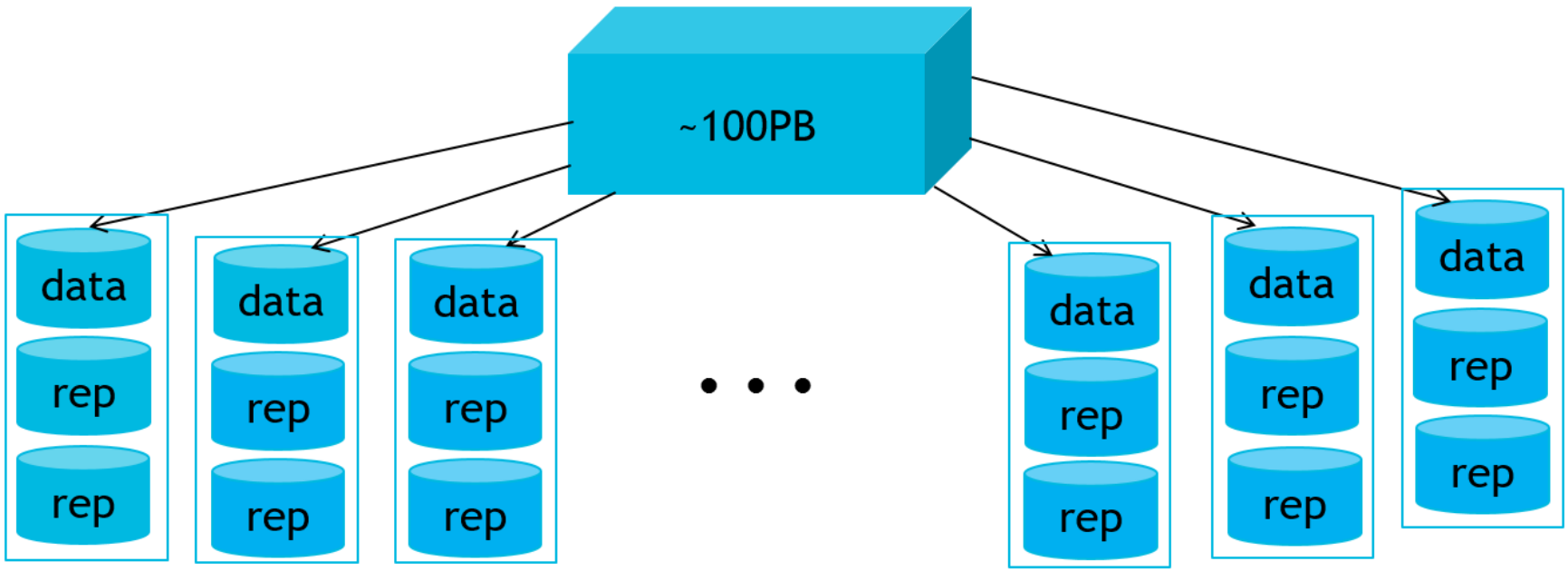


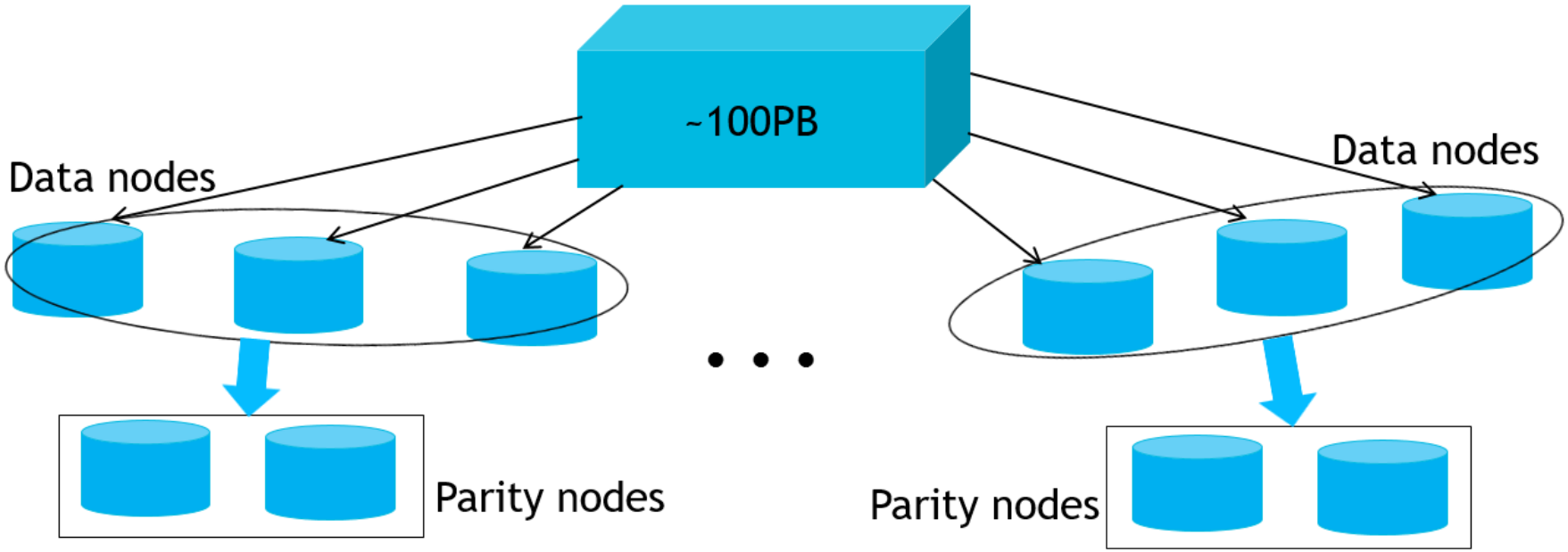
- DSS spread data across thousands of storage nodes
- Individual storage nodes fail frequently
- To protect the data we rely on erasure codes

Replication: large storage overhead



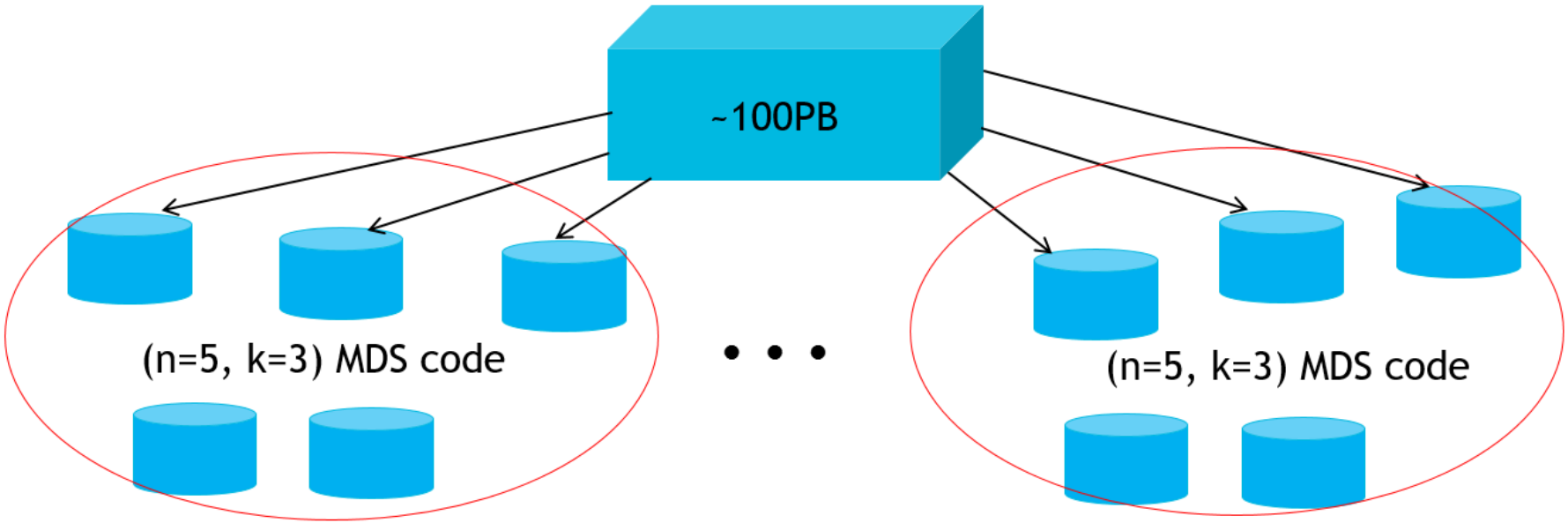
Can tolerate any 2 node failures
Storage overhead $= \quad 3\times$

MDS codes: Optimal storage efficiency



Add 2 parity nodes to every 3 data nodes
Form an $(n = 5, k = 3)$ MDS code

MDS codes: Optimal storage efficiency



Can tolerate any 2 node failures
Storage overhead $= \quad 1.67\times$

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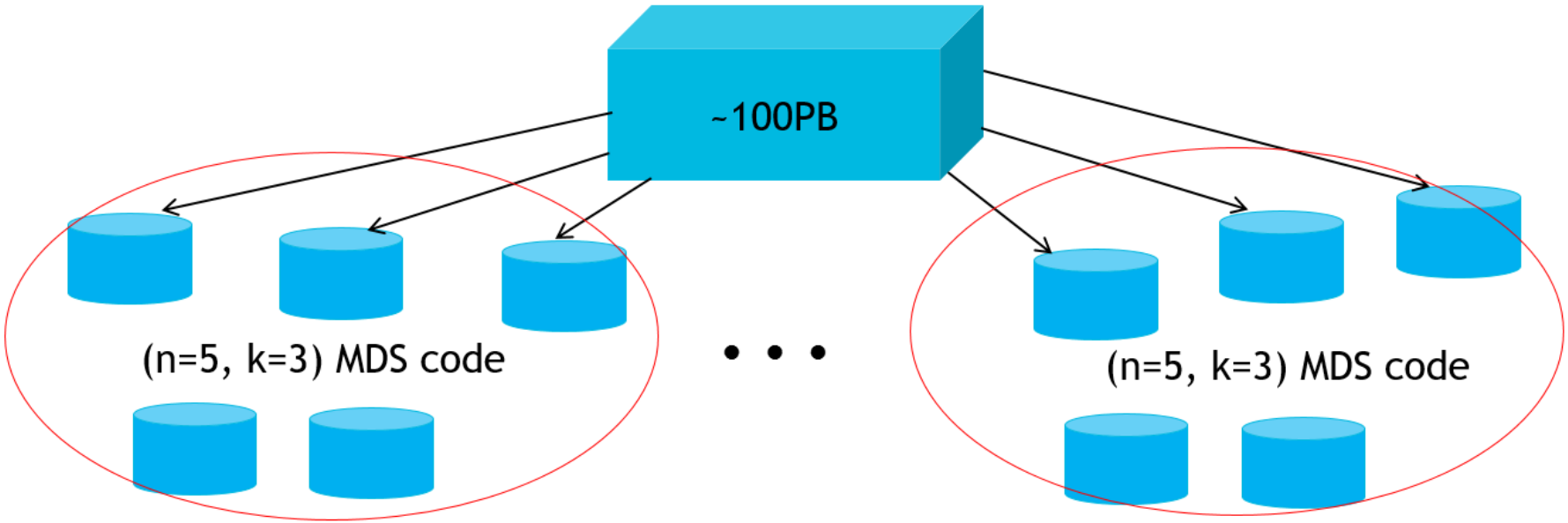
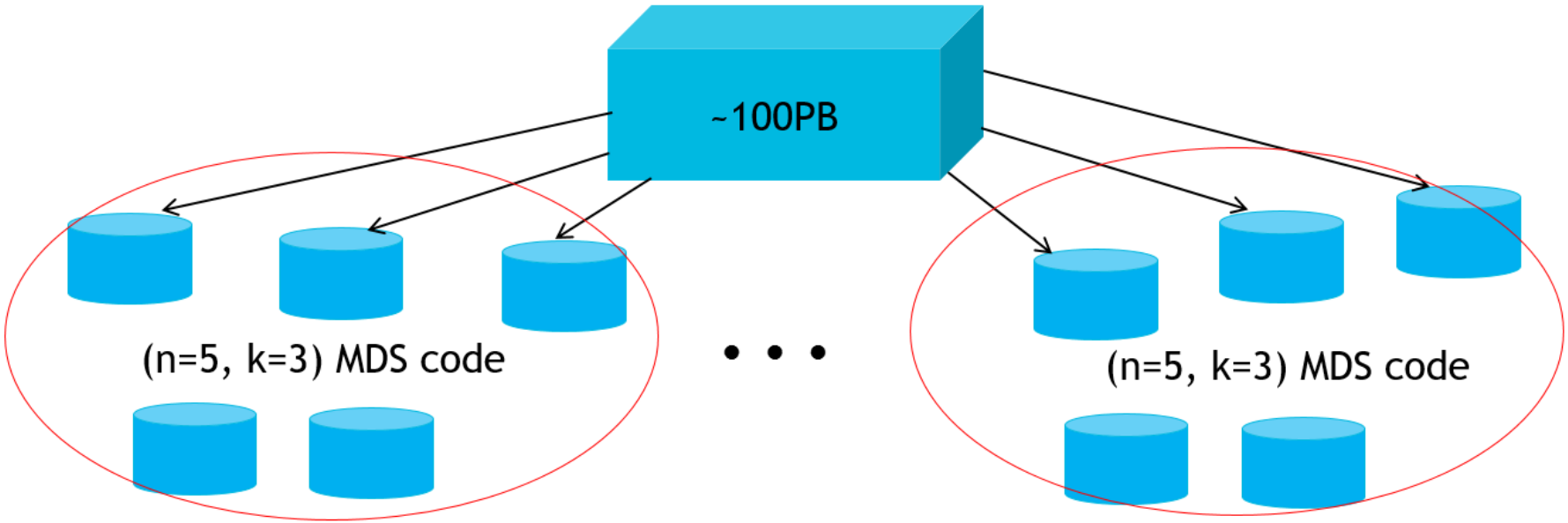


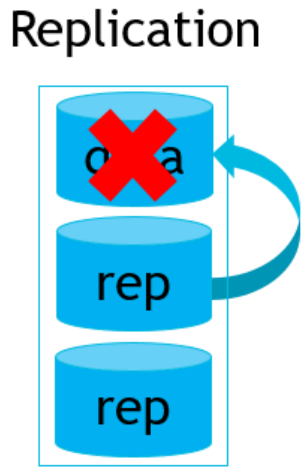
Block-based system model: Data blocks are encoded independently

MDS codes: Optimal storage efficiency

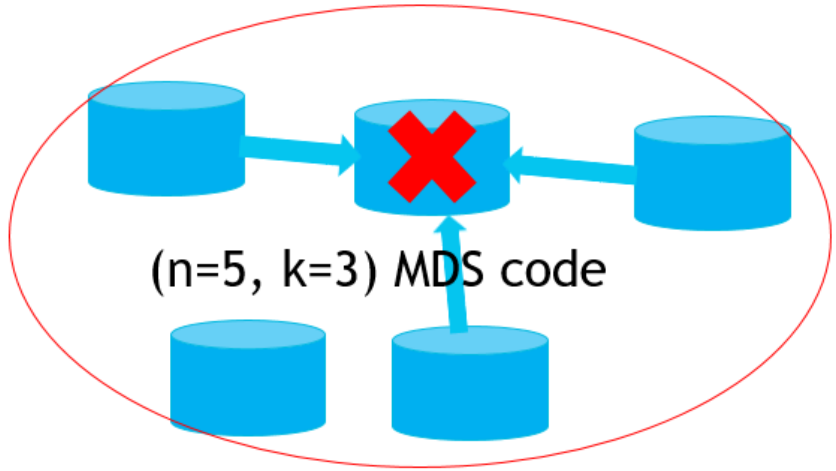


Each node stores l symbols of the block

Network flow incurred by data regeneration



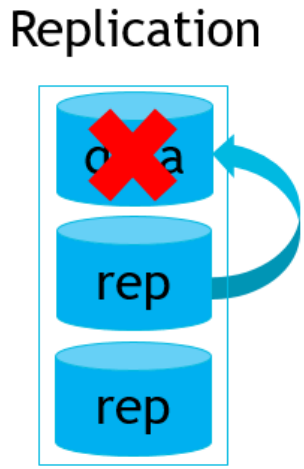
Network flow = node size



Network flow = 3 x node size

MDS code uses much more network bandwidth during data regeneration

Network flow incurred by data regeneration



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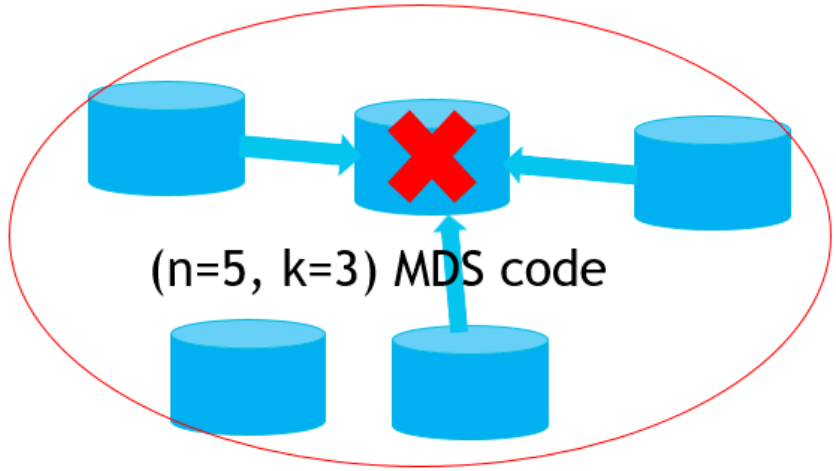

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MDS code uses much more network bandwidth during data regeneration

Goal: To find MDS codes using less bandwidth during data regeneration

Reed-Solomon codes

$F = \mathbb{F}_q$ the finite field of q elements

Example: $q = 8$, $\alpha^3 = \alpha + 1$

$\mathbb{F}_8 = \{0, 1, \alpha, \alpha^2, \alpha^3 = \alpha + 1, \alpha^4 = \alpha^2 + \alpha, \alpha^5 = \alpha^2 + \alpha + 1, \alpha^6 = \alpha^2 + 1\}$

The elements of $\mathbb{F}_8$ can also be written as binary vectors
$(000), (001), (010), (101), (110), (111), (101)$

Encoding example: Let $q = 8$, $(n, k) = (7, 3)$

Suppose that data symbols are $1, \alpha, \alpha$

To encode, form a polynomial $f(x) = 1 + \alpha x + \alpha x^2$ and compute the values of $f(x)$ at the points $1, \alpha, \alpha^2, \ldots, \alpha^6$

x	1	α	α^2	α^3	α^4	α^5	α^6
$f(x)$	1	α^4	α^6	α^4	α	α	α^6

This encodes $k = 3$ data symbols $(1, \alpha, \alpha)$ into $n = 7$ symbols of the codeword

Reed-Solomon codes

Reed-Solomon code $\mathcal{C}(A)$ over F encodes k data symbols into n *code symbols*

Let $f(x) = a_0 + a_1 x + \cdots + a_{k-1} x^{k-1}$ be the data (represented by polynomials) and encode as follows:

$$\begin{aligned} \mathcal{C} : F^k &\to F_q^n \\ f(x) &\mapsto (f(P_1), f(P_2), \ldots, f(P_n)) \end{aligned}$$

where $P_1, \ldots, P_n$ are some n elements of F

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In other words, $(c_1, \ldots, c_n)$ is an RS codeword if it satisfies the following ***parity-check equations:***

$$c_1 + c_2 \alpha^i + c_3 \alpha^{2i} \cdots + c_n \alpha^{(n-1)i} = 0, \quad i = 1, \ldots, n-k$$

Use of RS codes for node repair

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(a_0, a_1, a_2, a_3)

Use of RS codes for node repair

$$(a_0, a_1, a_2, a_3) \rightarrow f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$$

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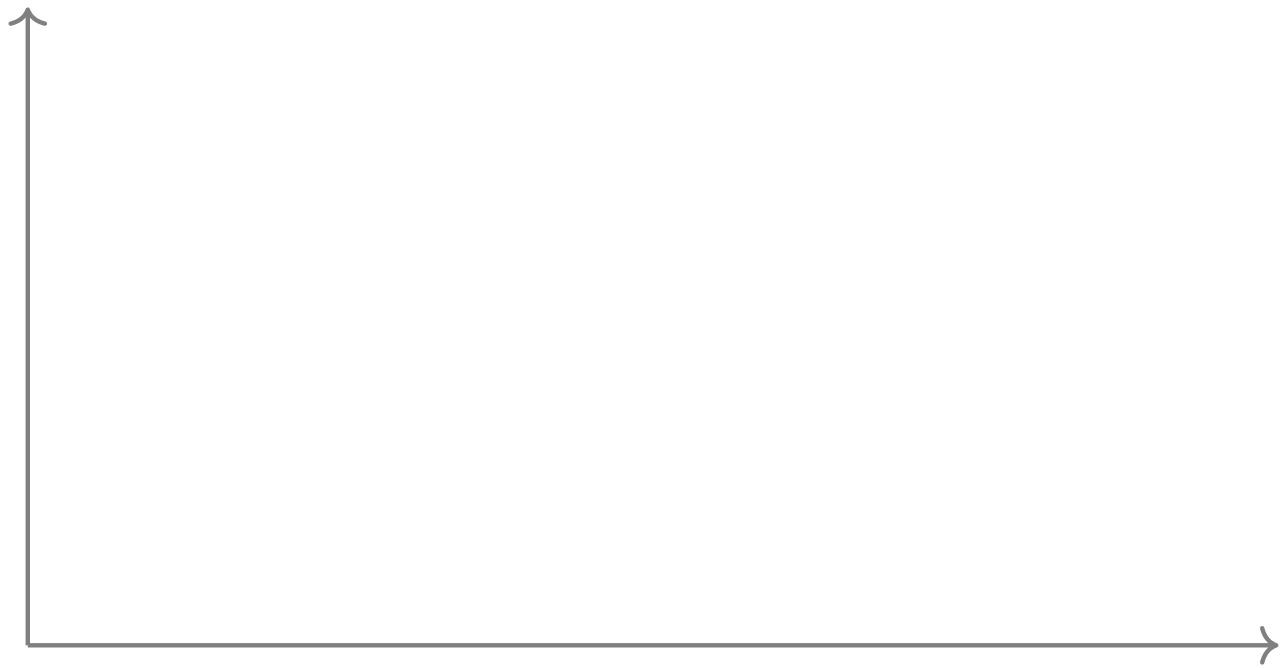

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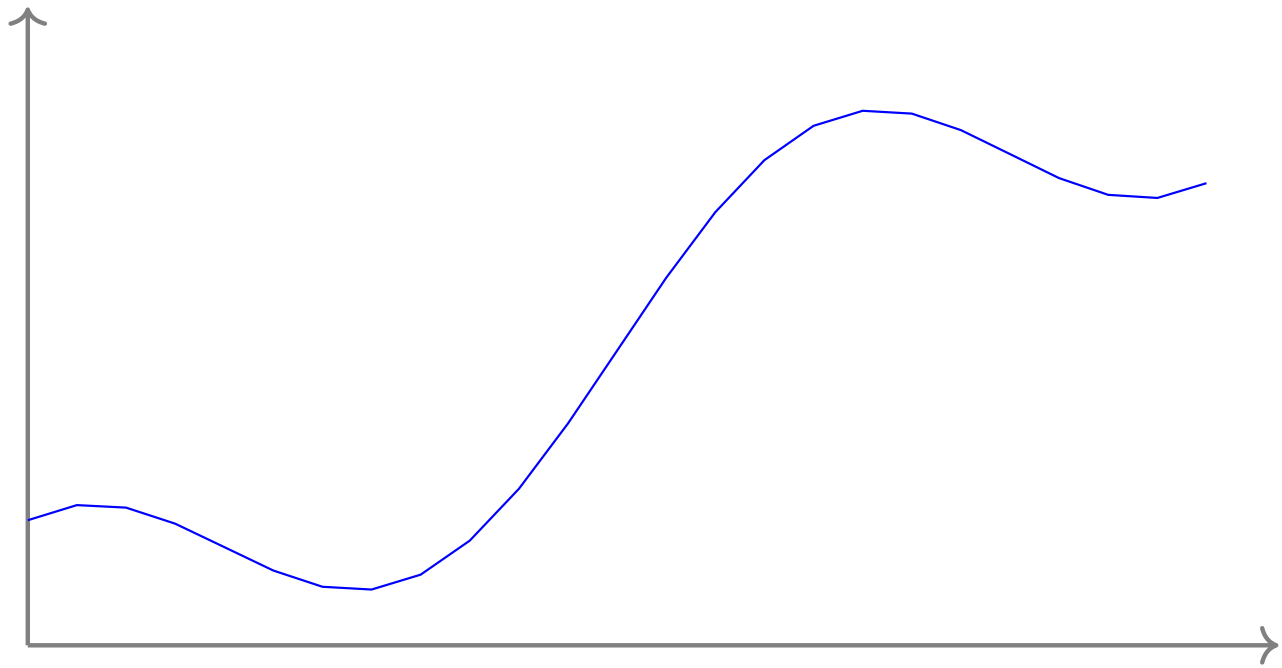

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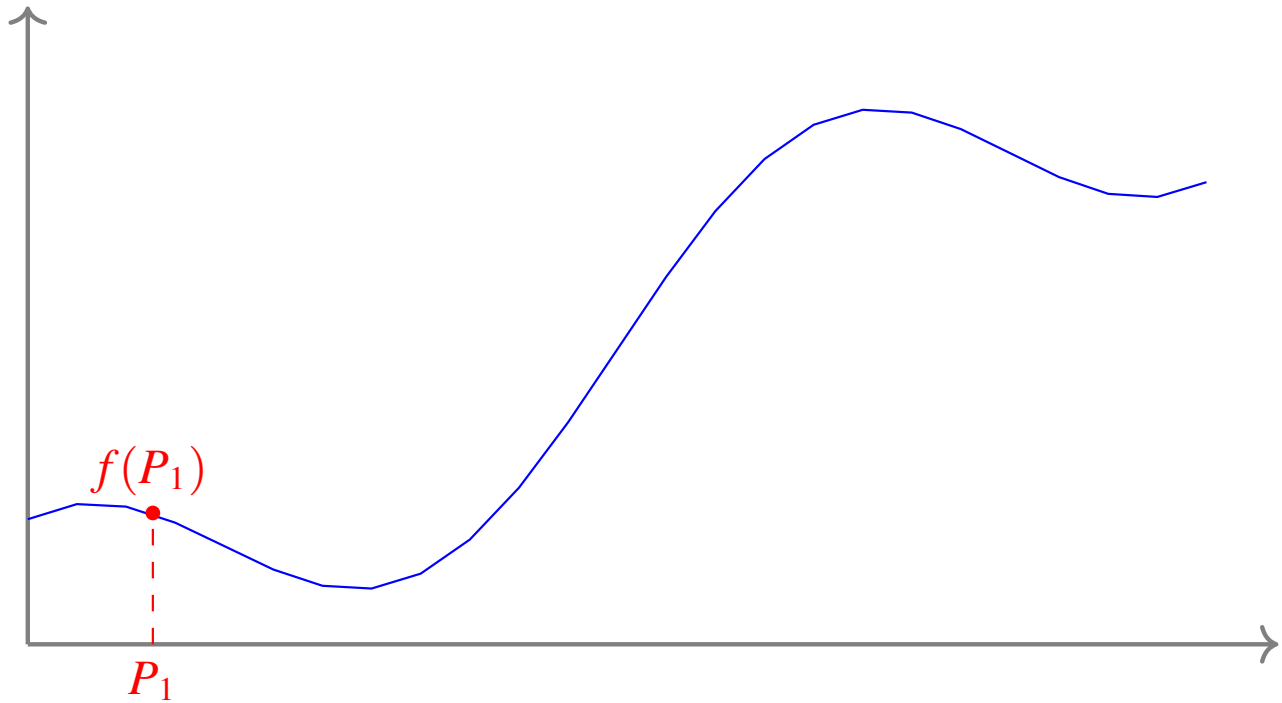

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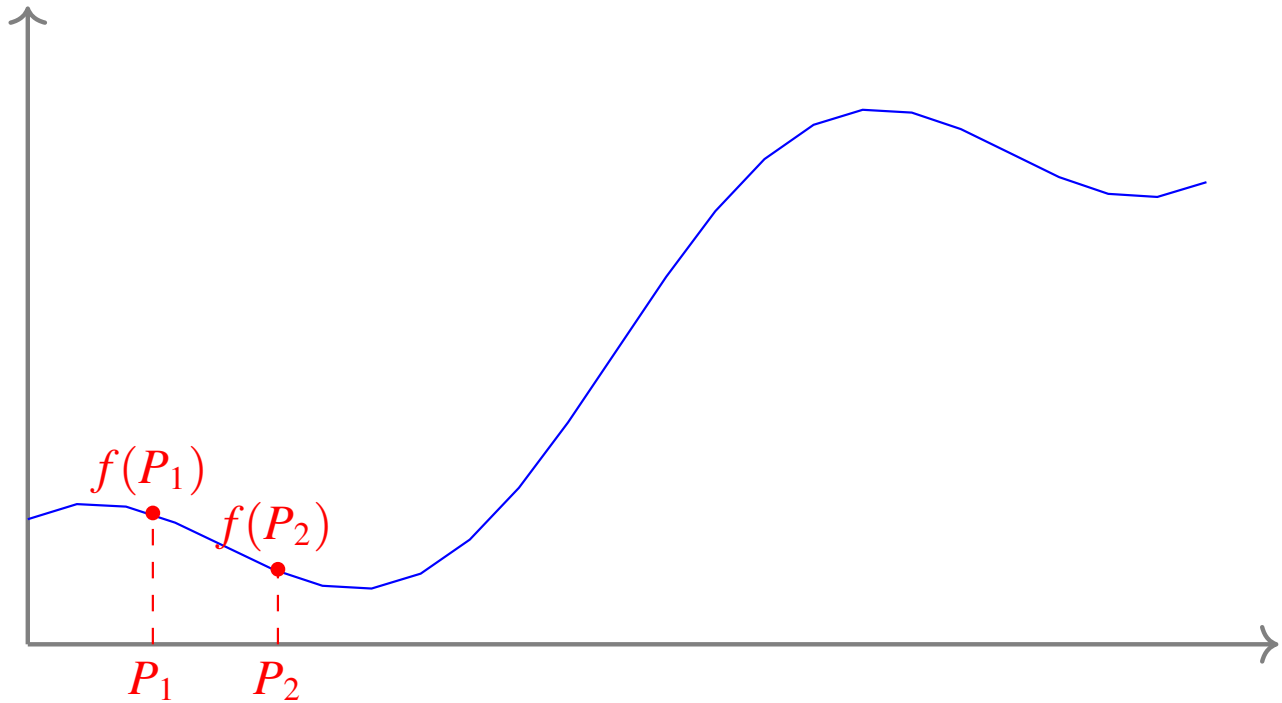

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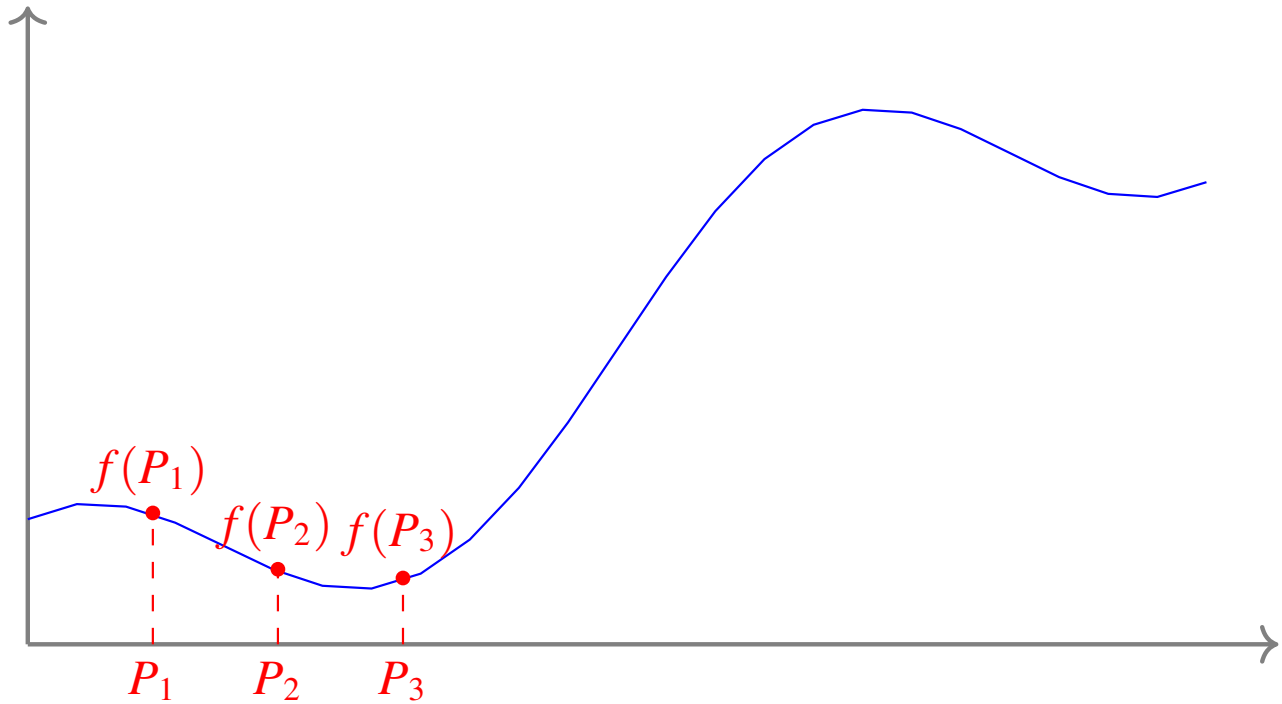

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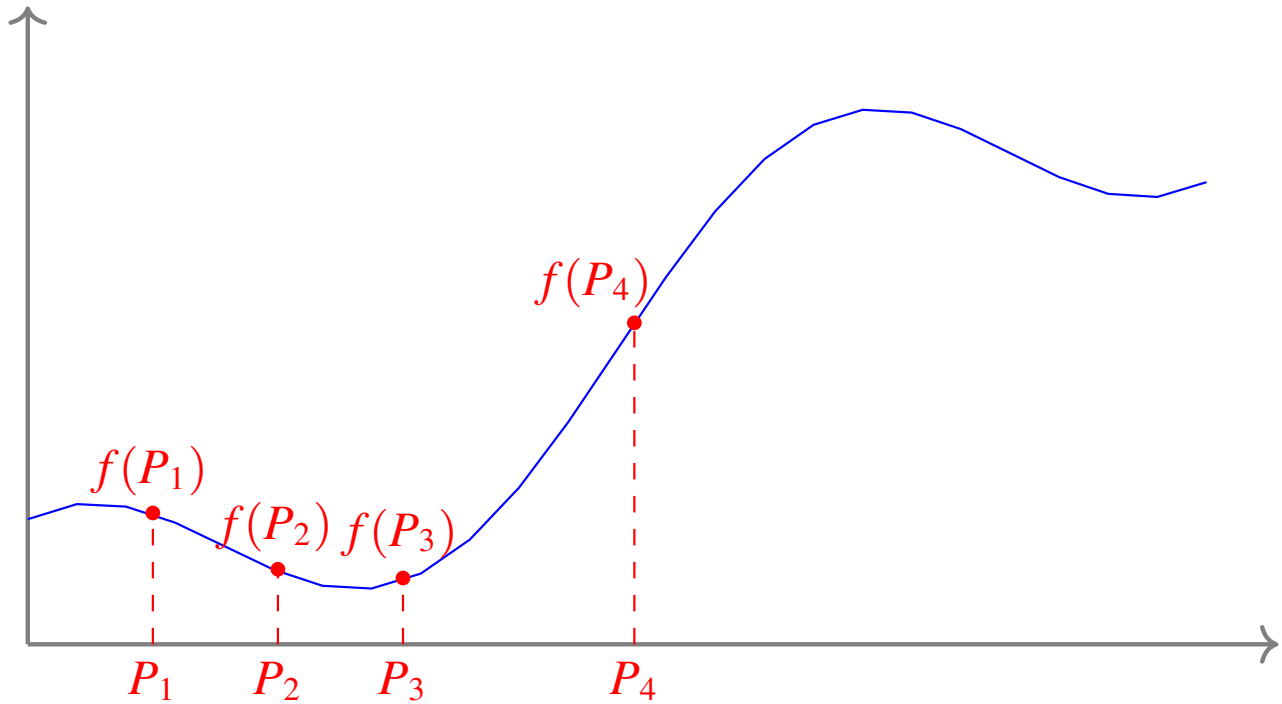

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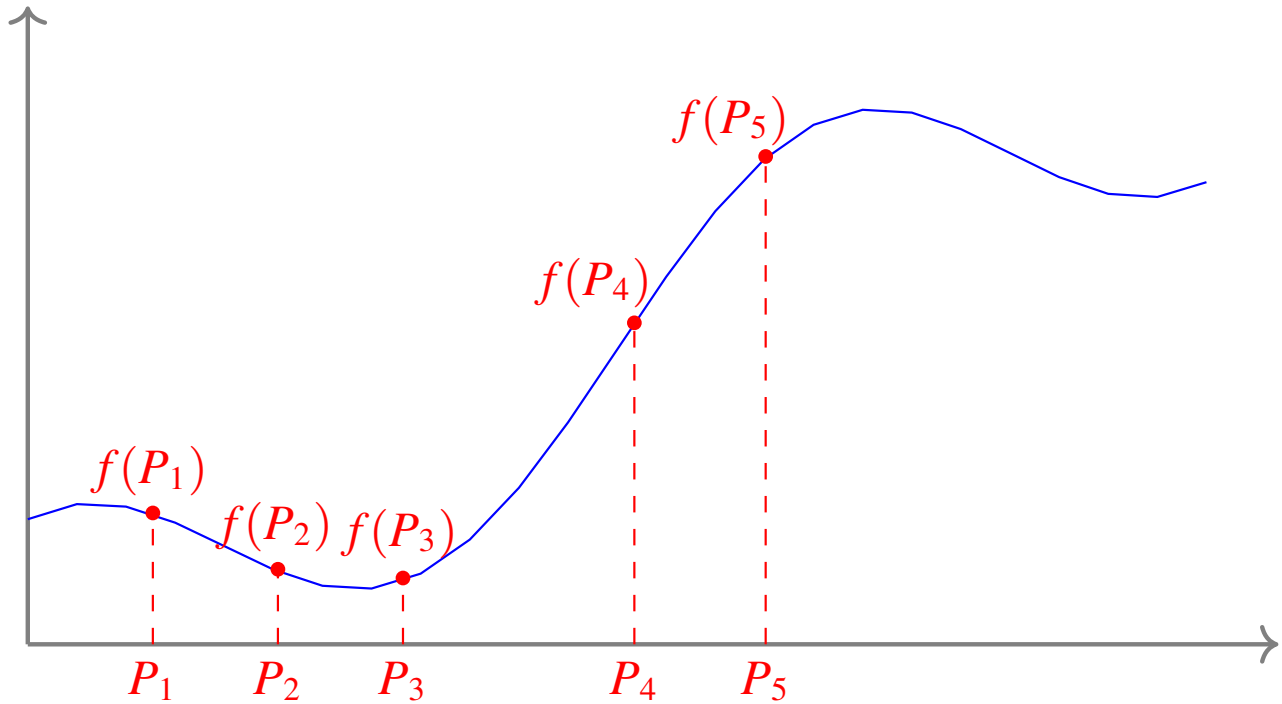

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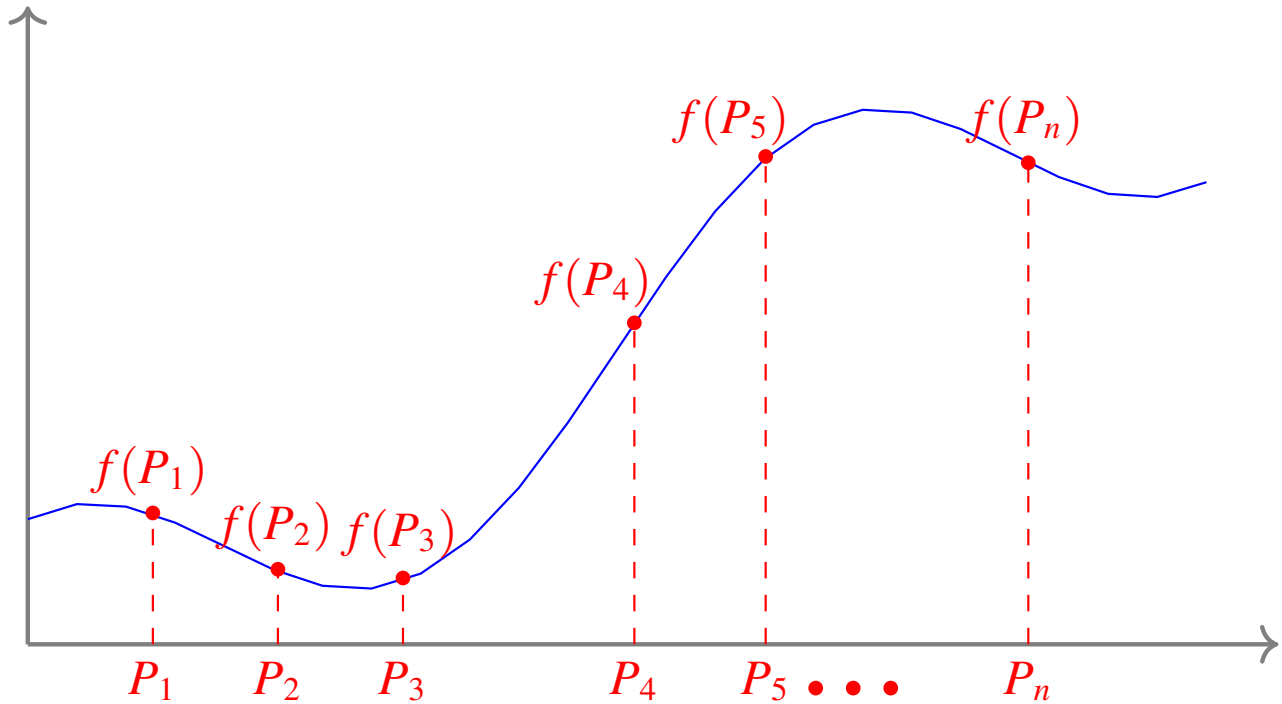

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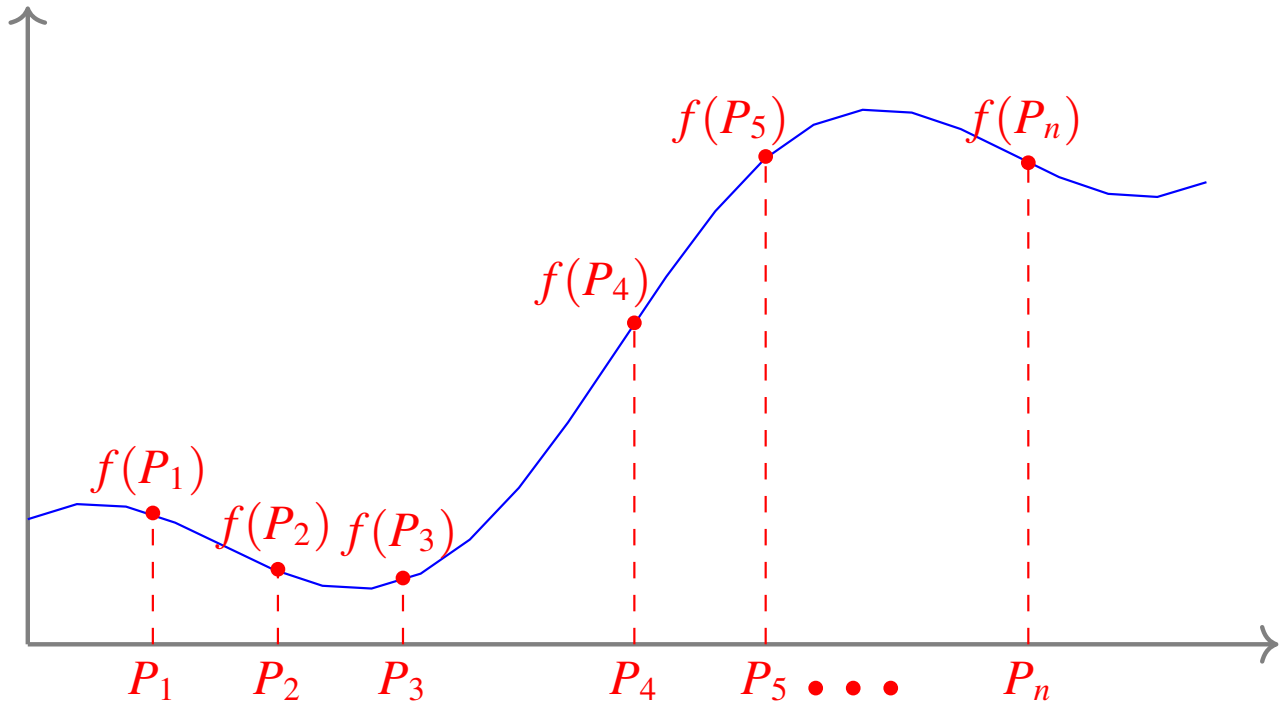

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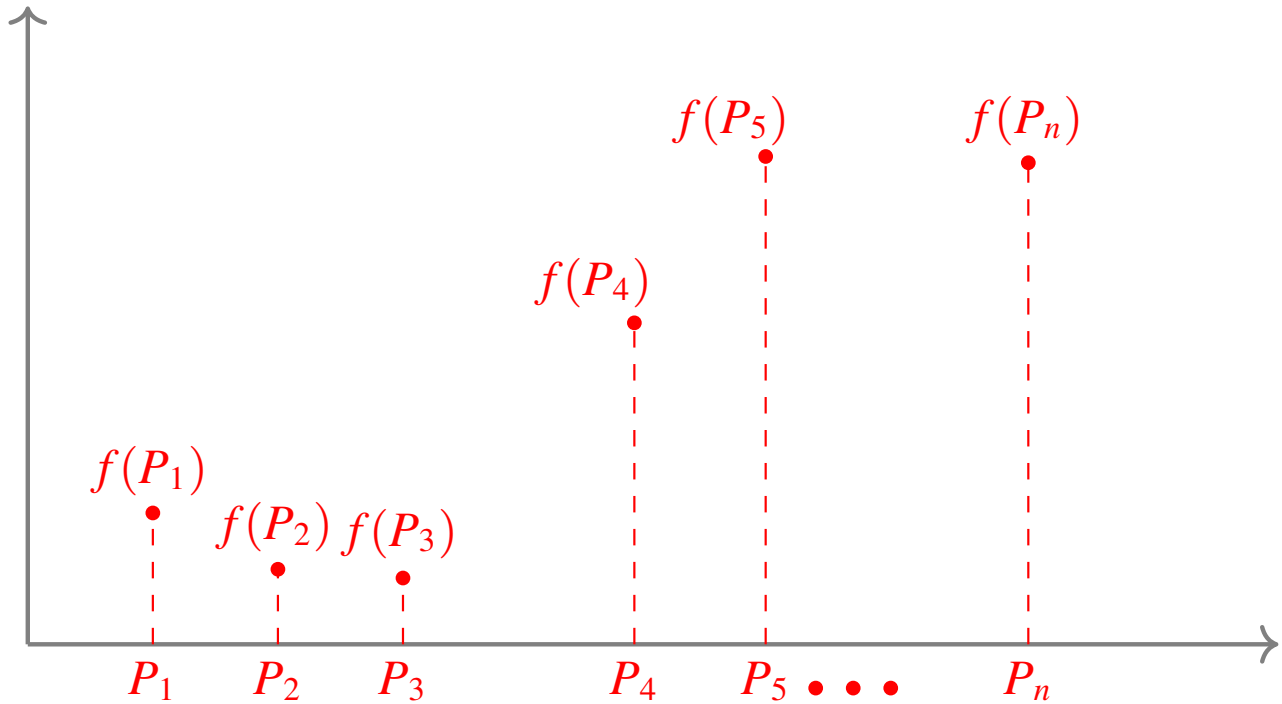

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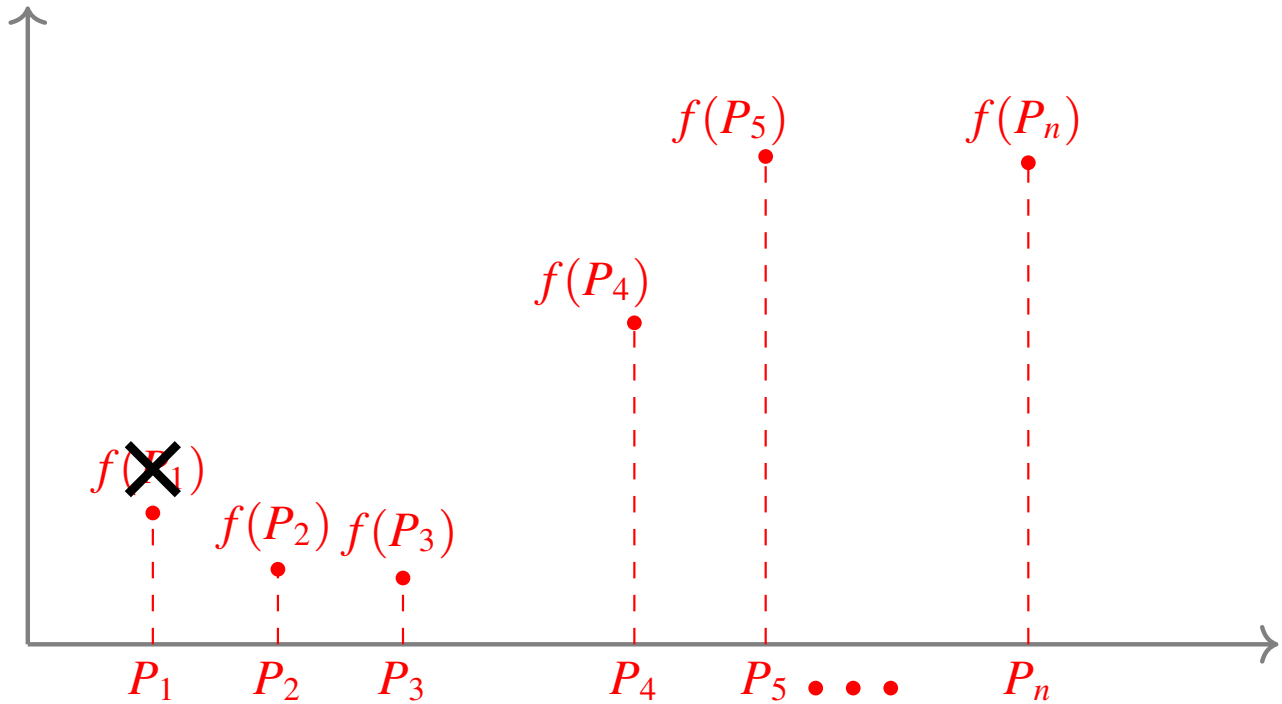

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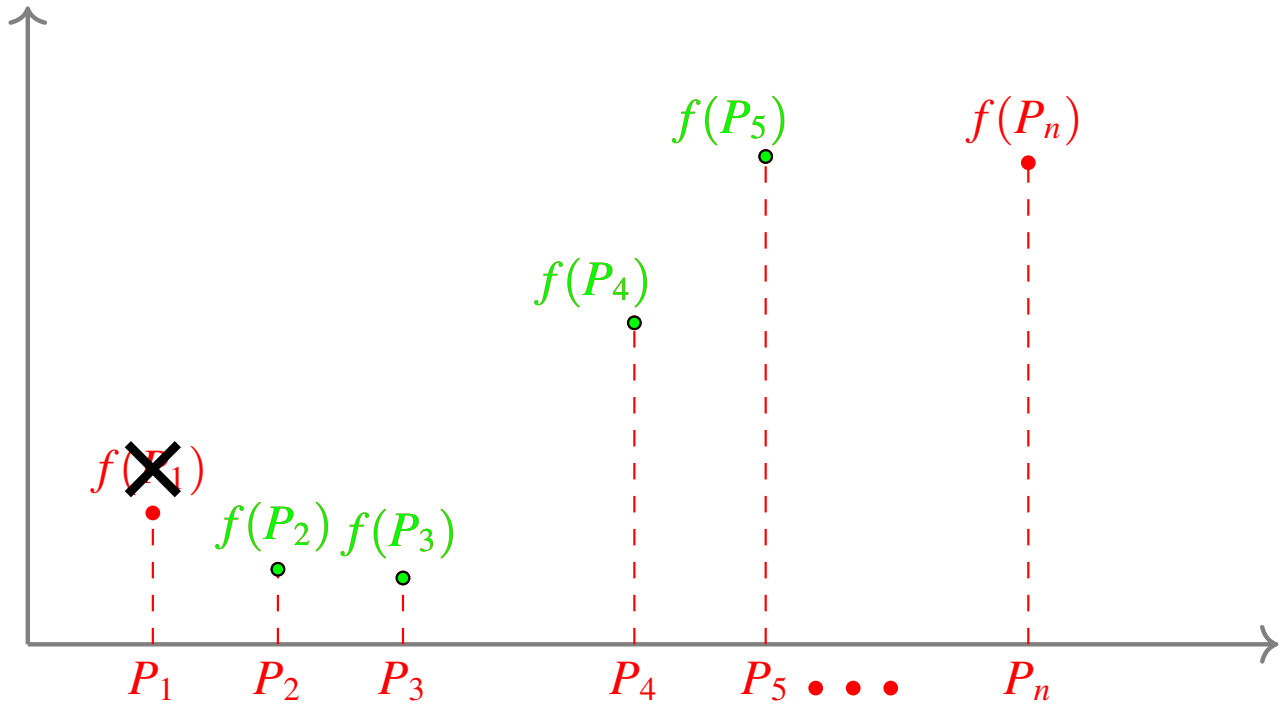

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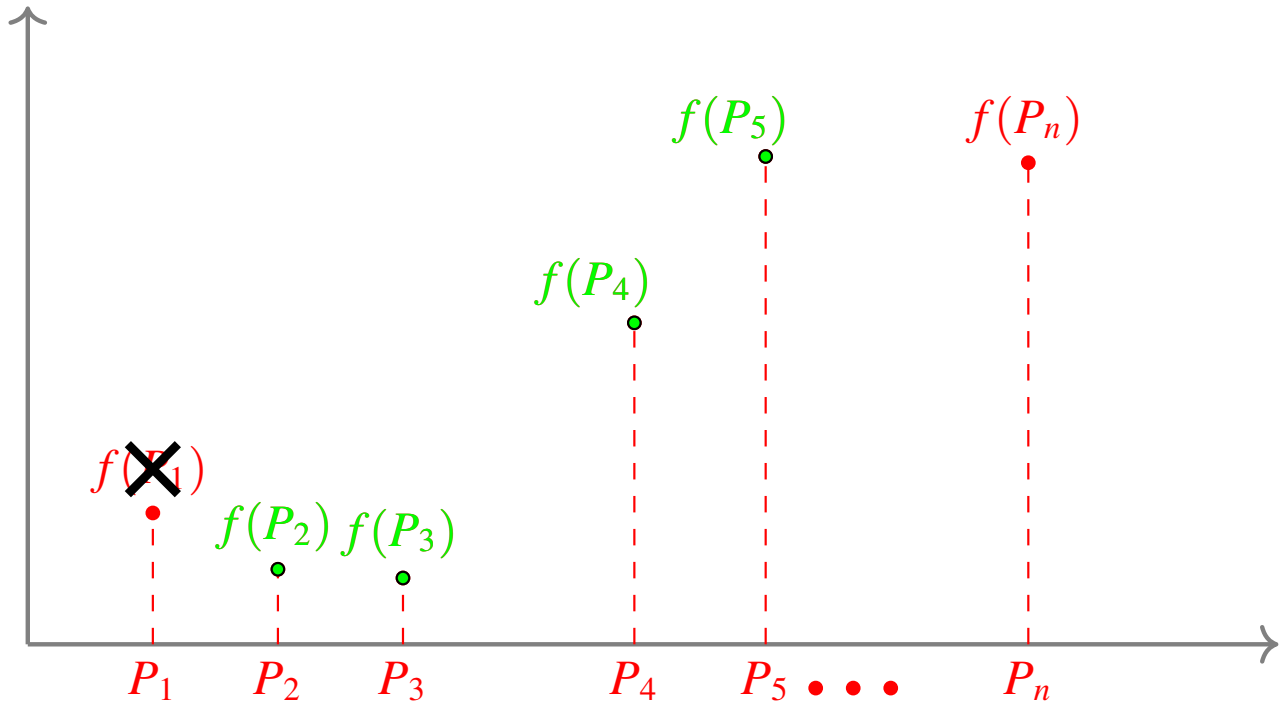

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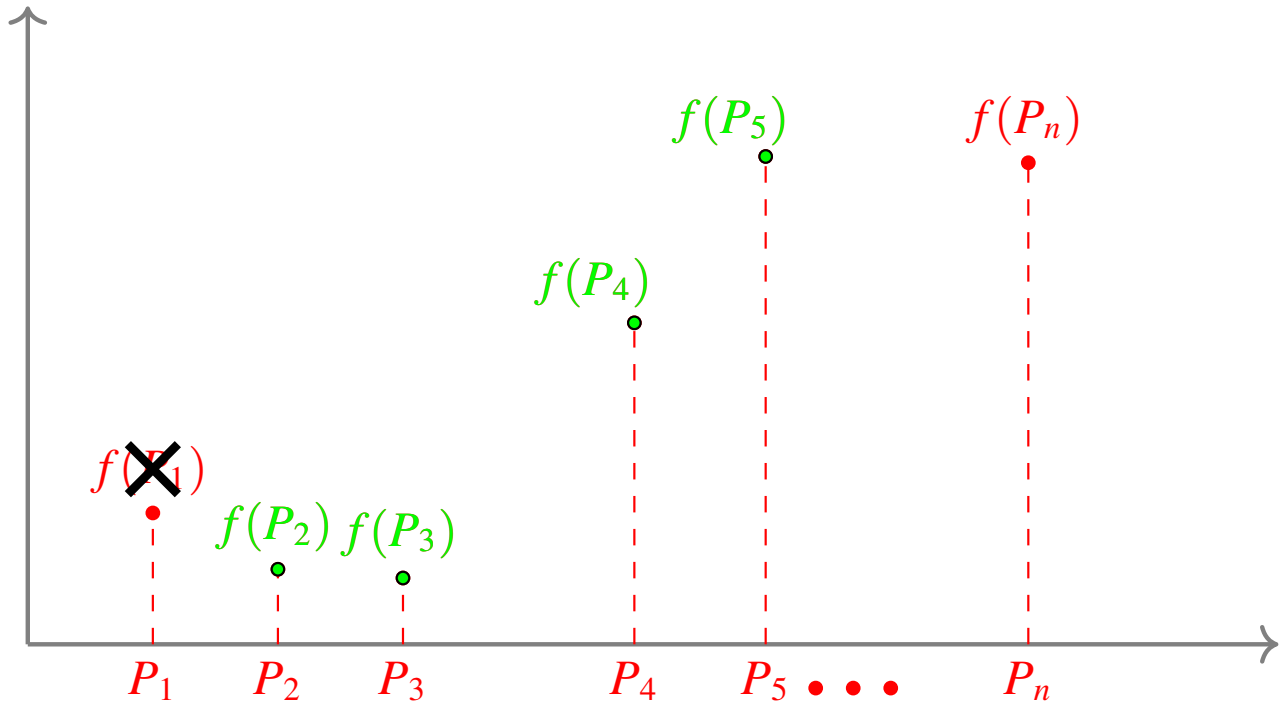
We need to download the contents of $k = 4$ nodes to repair a failed node



MDS array codes for storage

Main idea for improvement: Sub-packetize the encoding (consider vector codes over a finite field F)

[1] We use "coordinates" and "nodes" interchangeably

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Each coordinate[1] of the codeword $(C_1, C_2, \ldots, C_n) \in F^n$ is an l-dimensional vector over F, so the codeword can be viewed as an $l \times n$ array over F

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- (n, k, l) MDS array code:
  - code length n
  - k data nodes
  - $r = n - k$ parity nodes
  - each codeword coordinate is a vector of dimension l over $\mathbb{F}$

  - MDS property: Contents of any r nodes can be determined by the other k nodes.

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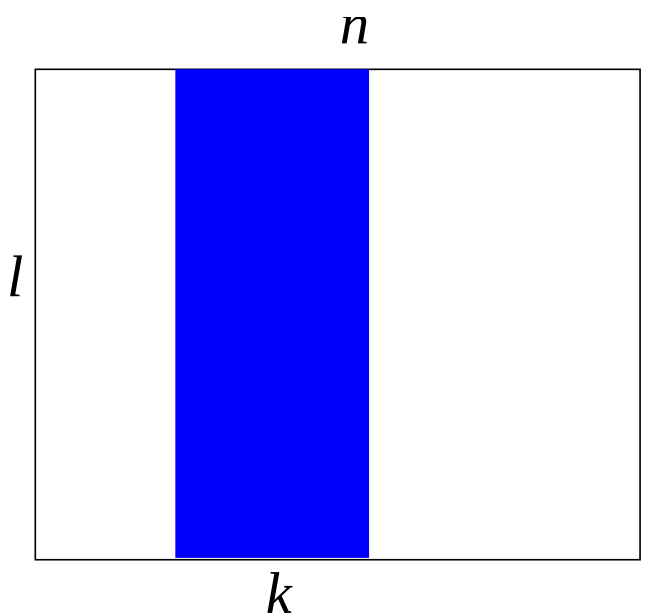
MDS array codes

Consider an $\ell \times n$ array code $\mathcal{C}$ ('vector code') over the field F



n - code length
k - code dimension, i.e., $|\mathcal{C}| = q^{k\ell}$
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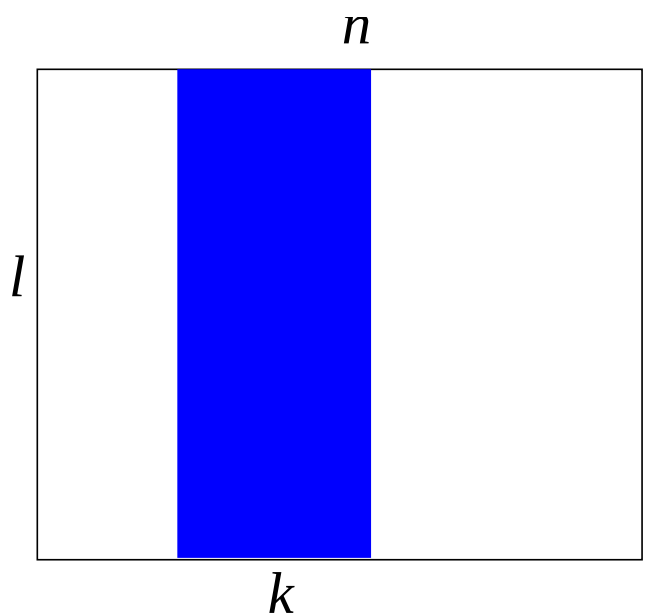

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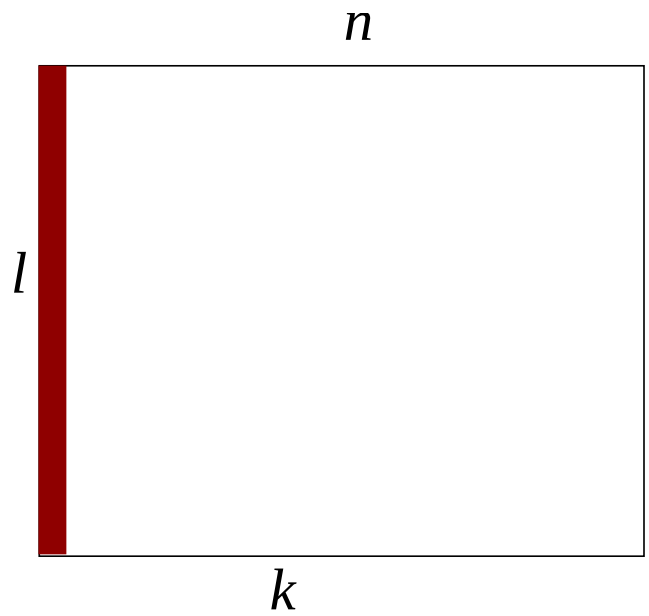
The code $\mathcal{C}$ is called MDS if any k columns suffice to recover the entire codeword
A column in this matrix is also called a node.

MDS array codes and repair bandwidth



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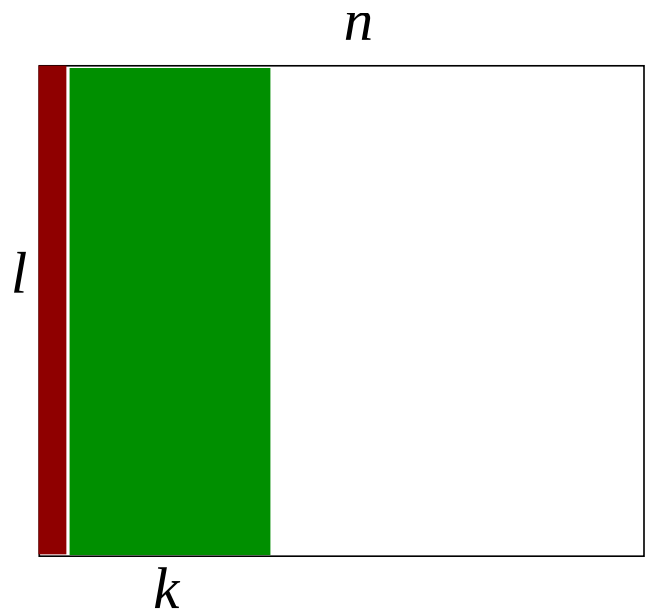

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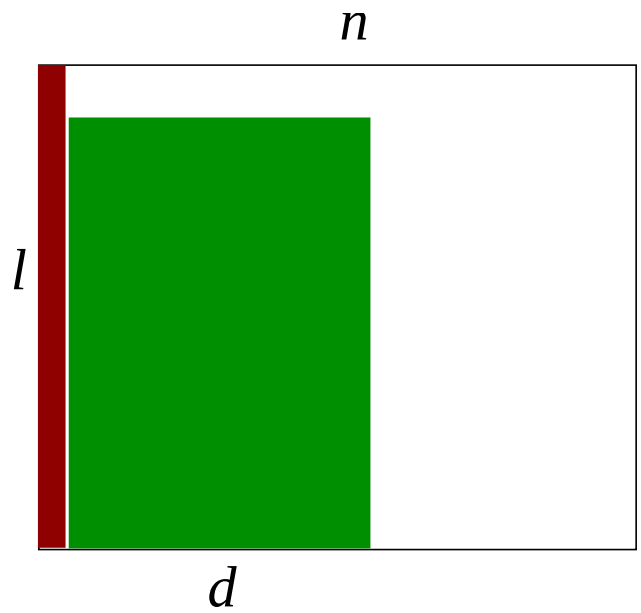

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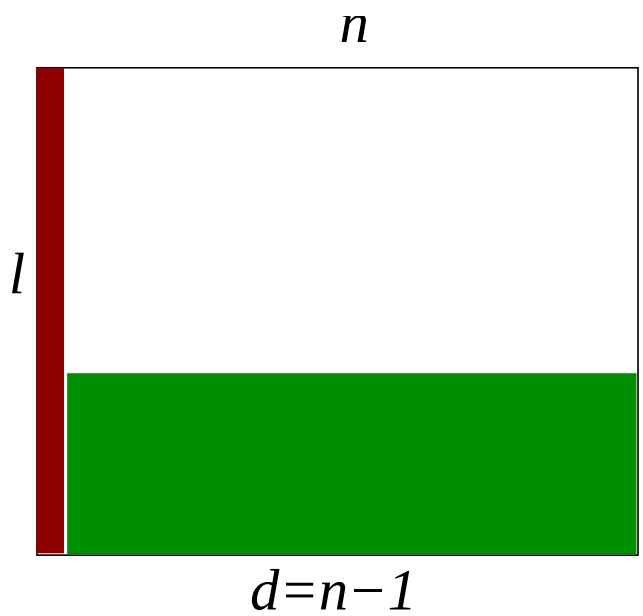

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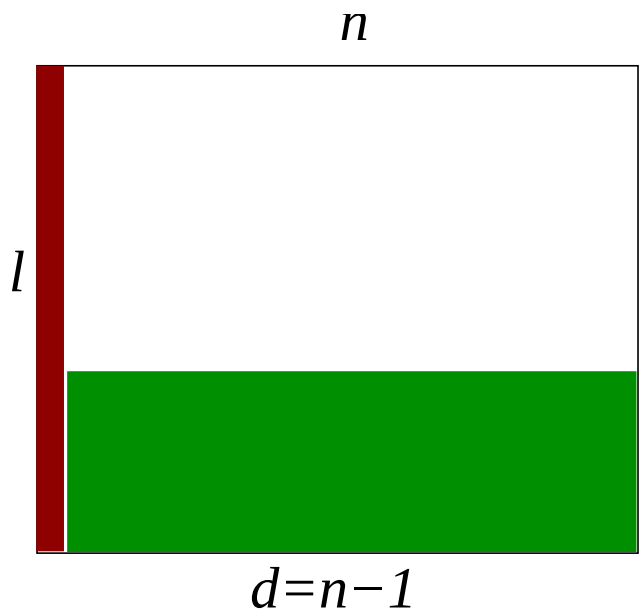

The number of 'downloaded' symbols is called the repair bandwidth β

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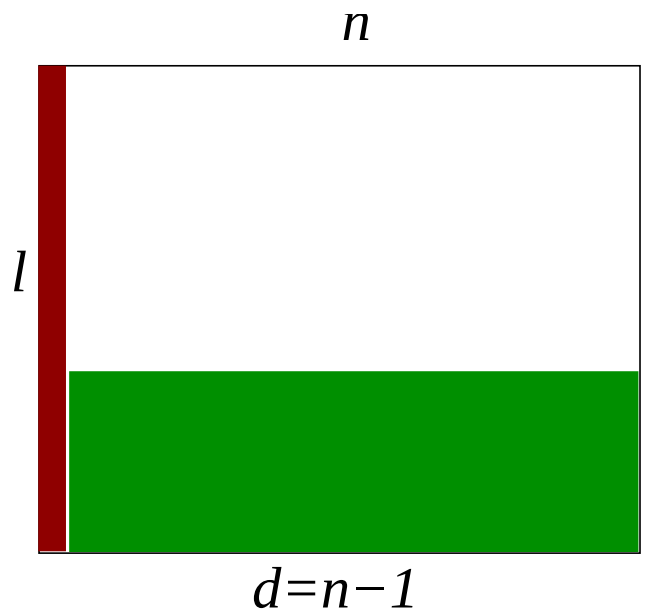

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$$\beta \geq \frac{d\ell}{d+1-k} \qquad \text{(Dimakis et al., 2010)}$$

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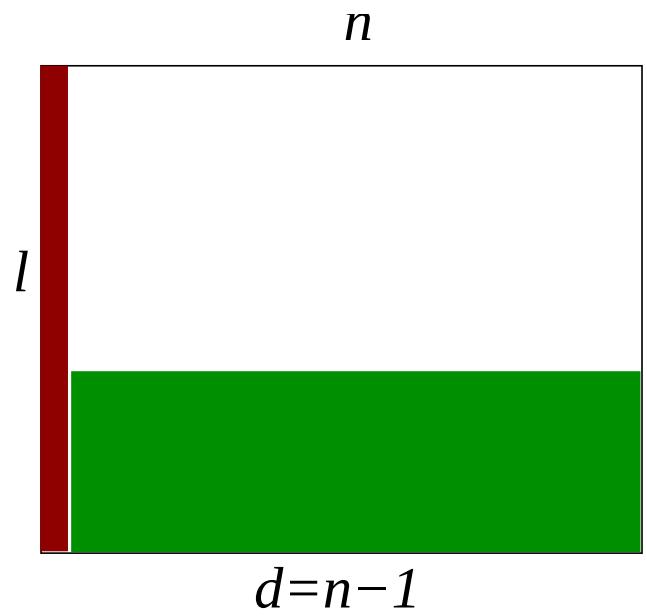

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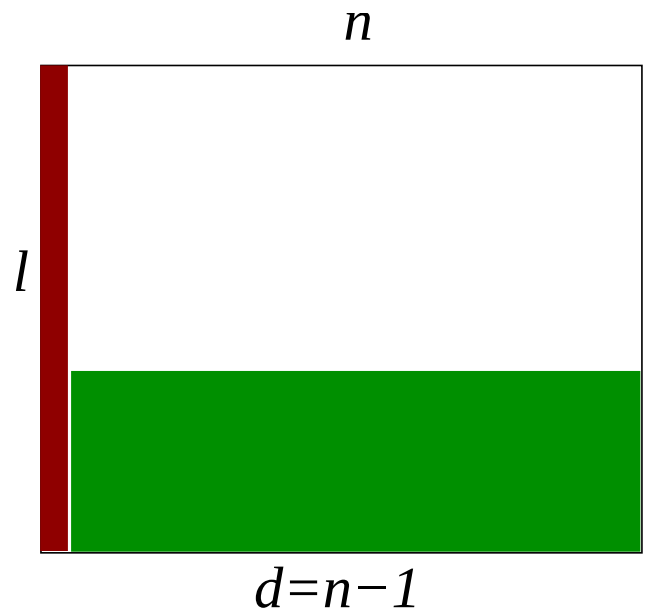
The right-hand side is a decreasing function of d, thus $\beta \to \min$ if $d = n-1$

MDS array codes and repair bandwidth



The number of 'downloaded' symbols is called the repair bandwidth β

$$\beta \geq \frac{d\ell}{d+1-k} \qquad \text{(Dimakis et al., 2010)}$$

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The code meeting this bound with equality is said to have optimal repair bandwidth

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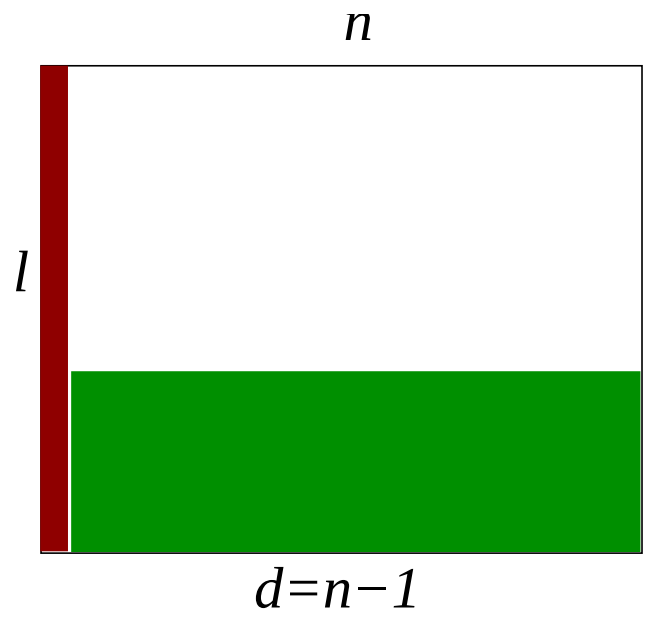

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For $d = n-1$, $r = n-k$

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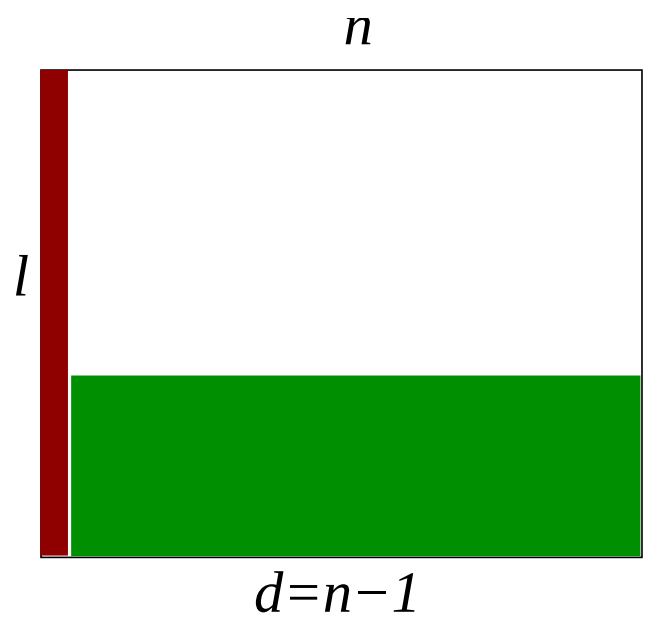
The repair problem



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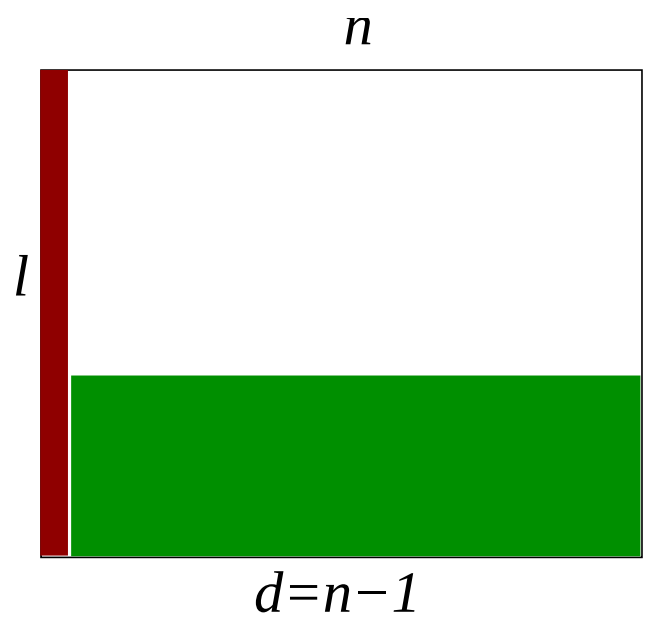

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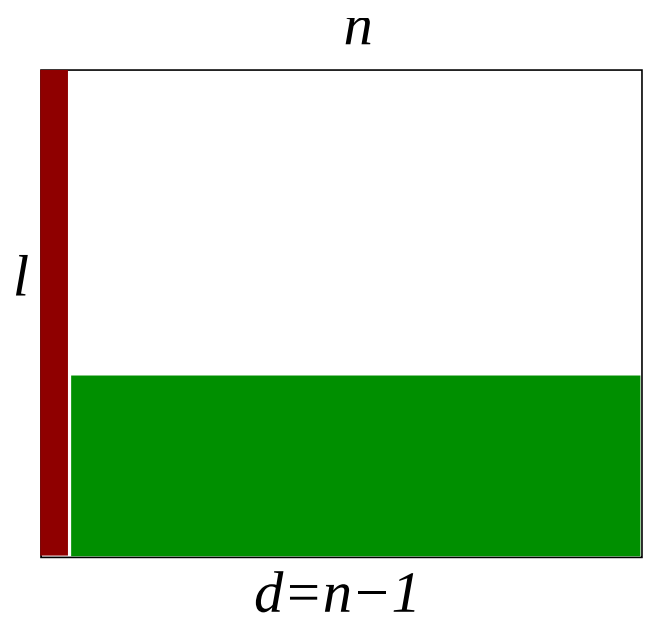

- The contents of the erased column is a function of the contents of the helper nodes.
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- If in addition the code has optimal repair bandwidth, it is called the optimal-access code

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The codes that meet this bound are said to have (h,d)-*optimal repair property*

This talk

Part I: Explicit constructions of MDS array codes with optimal repair

Part II: Asymptotically optimal family of Reed-Solomon codes

Part I: Optimal-repair MDS array codes

Previous results (a sampling):

- Low rate $k/n \leq 1/2$: Constructions of codes with $(1, n-1)$ optimal repair (RASHMI ET AL., '11, SUH-RAMCHANDRAN, '11)
- High rate: Constructions for $r = 2, 3$ (TAMO-WANG-BRUCK '13; PAPAILIOPOULOS ET AL. '13; WANG-TAMO-BRUCK '16; RAVIV-SILBERSTEIN-ETZION '17)
- High rate, any r: A number of existence results over large alphabets (CADAMBE ET AL., '12; WANG ET AL., '16; GOPARAJU-FAZELI-VARDY, '16)
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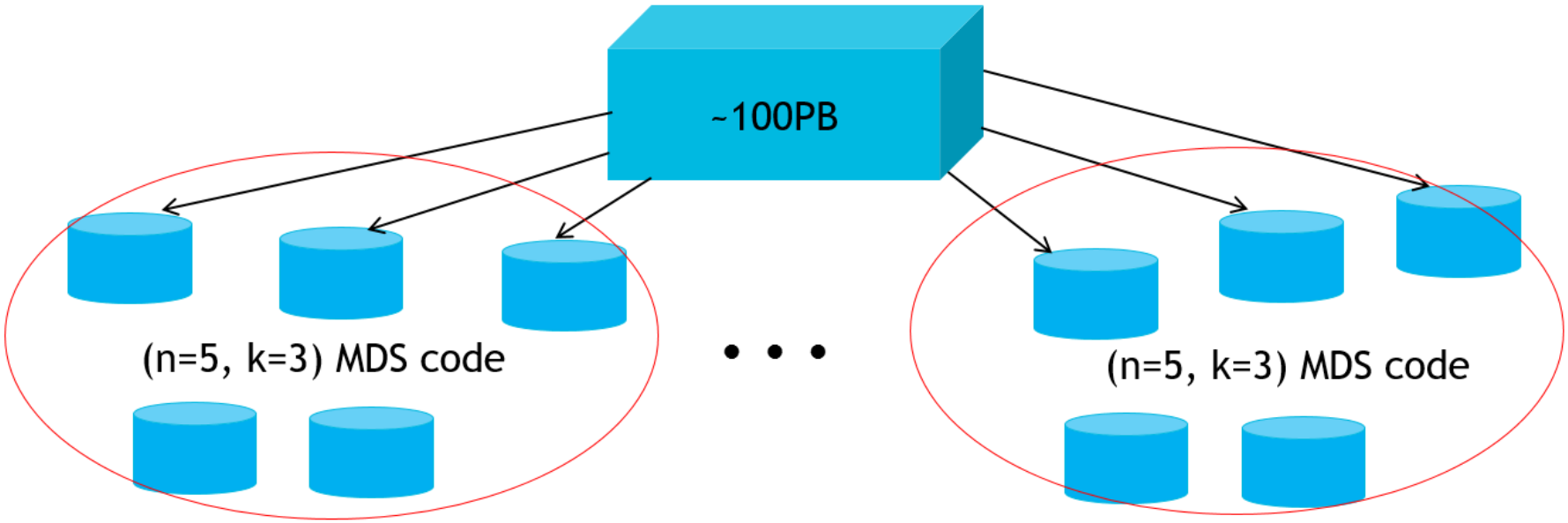
In this talk (MIN YE and A.B., '16-'17):

- A construction of (h, d)-optimal repair codes for any r, any $h \leq r$, and $d, k \leq d \leq n - h$
- Code families that work for all h and all d simultaneously
- Error-resilient optimal-repair codes
- Optimal-access codes with the above properties

Placement of MDS codewords in storage nodes



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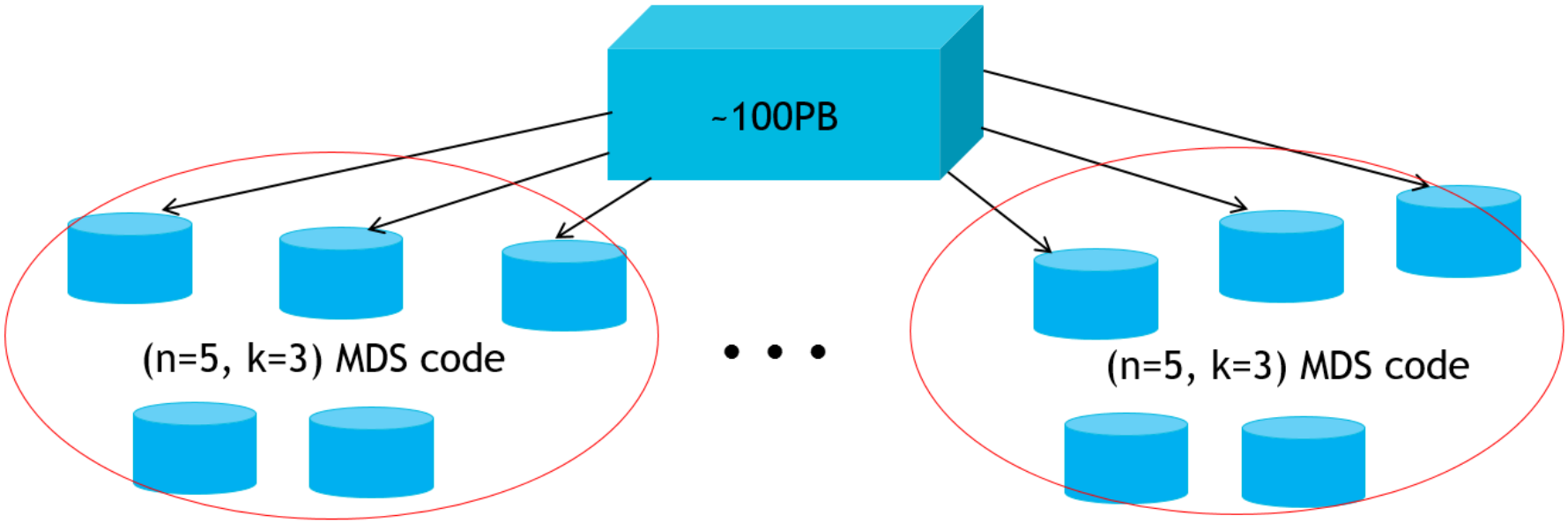

Any k nodes suffice to recover the data

Each node stores l symbols of the block

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- $r \times n$ parity check matrix

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- Natural choice: Diagonal matrices

Construction I: Optimal $(1, n-1)$ repair MDS codes

- Take $l = r^n$; take the $l \times l$ matrix $A_i, i = 1, \ldots, n$ in the form

$$A_i = \sum_{a=0}^{l-1} \lambda_{i,a_i} e_a e_a^T$$

where λ_{ij} are distinct elements of F and $(a_n, a_{n-1}, \ldots, a_1)$ is r-ary expansion of a

[1] Note that we need $r \cdot n$ distinct elements of the finite field F, i.e., $|F| \geq rn$.

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- The codeword has the form $C = (C_1, \ldots, C_n)$, where $C_i = (c_{i,0}, c_{i,1}, \ldots, c_{i,l-1})^T$

$$\begin{matrix} c_{1,0} & c_{2,0} & \ldots & c_{n,0} \\ c_{1,1} & c_{2,1} & \ldots & c_{n,1} \\ \vdots & \vdots & \vdots & \vdots \\ c_{1,l-1} & c_{2,l-1} & \ldots & c_{n,l-1} \end{matrix}$$

- Idea: Every row forms an RS code with different evaluation points $\{P_{i,j}\}$
  For $a = 0, 1, \ldots, l-1$, write r-ary expansion $a = (a_1, a_2, \ldots, a_n)$
  Evaluation points for a-th row: $(\lambda_{1,a_1}, \lambda_{2,a_2}, \ldots, \lambda_{n,a_n})$

$(1, n-1)$-optimal repair property

Idea (cont'd): Let $C_i = (c_{i,a}, a = 0, 1, \ldots, l-1)$ be the missing node.
Repair the contents by ***groups of size*** r that differ only in position i of the expansion

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$$\sum_{u=0}^{r-1} \lambda_{i,u}^{t} c_{i,a(i,u)} + \sum_{j \neq i} \Big(\sum_{u=0}^{r-1} \lambda_{j,a_j}^{t} c_{j,a(i,u)} \Big) = 0, \qquad t = 0, 1, \ldots, r-1$$

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$$\sum_{u=0}^{r-1} c_{j,a(i,u)}$$

$$+ \sum_{j \neq i} \underbrace{\begin{bmatrix} 1 & 1 & \cdots & 1 \\ \lambda_{j,a_j} & \lambda_{j,a_j} & \cdots & \lambda_{j,a_j} \\ \vdots & \vdots & \vdots & \vdots \\ \lambda_{j,a_j}^{r-1} & \lambda_{j,a_j}^{r-1} & \cdots & \lambda_{j,a_j}^{r-1} \end{bmatrix}}_{\text{aligned, rank } = 1} \begin{bmatrix} c_{j,a(i,0)} \\ c_{j,a(i,1)} \\ \vdots \\ c_{j,a(i,r-1)} \end{bmatrix} = 0$$

$(1, n-1)$-optimal repair property

- For $u = 0, 1, \ldots, r-1$ let $a(i,u) := (a_n, \ldots, a_{i+1}, u, a_{i-1}, \ldots, a_1)$.
- Partition the symbols on the failed node i into r^{l-1} groups of size r each:

$$\{c_{i,a(i,0)}, c_{i,a(i,1)}, \ldots, c_{i,a(i,r-1)}\}$$

  for some $a \in \{0, 1, \ldots, l-1\}$

- Say C_i is unavailable. The elements in each group can be found by acquiring ***one element*** $\sum_{u=0}^{r-1} c_{ja(i,u)}$ from each of the $n-1$ remaining nodes.

- Total repair bandwidth $= (n-1) \times 1 \times r^{l-1} = (n-1)(l/r)$, matching the lower bound

Construction II: Codes with $(1, d)$-optimal repair and error correction, $k < d < n - 1$

The bound on the repair bandwidth can be extended to error correction:

$$\beta(d, t) \geq \frac{(d + 2t)l}{d + 1 - k}$$

(PAWAR ET AL., '11; RASHMI ET AL., '12)

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- Construct (n, k) MDS code $\mathcal{C}$ over $(F)^l$, $l = s^n$, $s = d + 1 - k$
- Take $|F| \geq sn$, $\{\lambda_{ij}, i = 1, \ldots, n; j = 0, \ldots, s - 1\} \subset F$,
- Parity-check matrix H is a block Vandermonde matrix with

$$A_i = \sum_{a=0}^{l-1} \lambda_{i,a_i} e_a e_a^T$$

  where $a = (a_n, a_{n-1}, \ldots, a_1)$ is the s-ary expansion of $a = 0, 1, \ldots, l - 1$.
- We use the fact that the symbols transmitted from the helper nodes form a (generalized) RS code

Construction III: Codes with (h, d)-optimal repair for all h, d simultaneously

We take $s = \text{lcm}(1, 2, \ldots, r)$

Sequential repair:

Let $C_1, \ldots, C_h$ be the failed nodes, $\mathcal{R}, |\mathcal{R}| = d + 2t$ be the helper set.

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- $C_{\mathcal{R}} \cup C_1 \cup C_2 \cup \cdots \cup C_{h-1}$ to repair C_h

Further extensions

1. d-optimal repair $(n, k = n - r, l = r^{n-1})$ MDS codes with optimal access property and error correction, $|F| \geq n + 1$
   (Idea: Use permutation matrices instead of diagonal matrices)

2. Codes for optimal repair from $d_1, \ldots, d_m$ helper nodes simultaneously, $k \leq d_1, \ldots, d_m < n$

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The codes have small encoding, repair, and update complexity.

They also afford efficient *systematic representation*

Systematic encoding; Updates

For $a = 0, 1, \ldots, l-1$

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ \lambda_{k+1,a_{k+1}} & \lambda_{k+2,a_{k+2}} & \cdots & \lambda_{k+r,a_{k+r}} \\ \vdots & \vdots & \vdots & \vdots \\ \lambda_{k+1,a_{k+1}}^{r-1} & \lambda_{k+2,a_{k+2}}^{r-1} & \cdots & \lambda_{k+r,a_{k+r}}^{r-1} \end{bmatrix} \begin{bmatrix} c_{k+1,a} \\ c_{k+2,a} \\ \vdots \\ c_{k+r,a} \end{bmatrix}$$

$$= - \begin{bmatrix} 1 & 1 & \cdots & 1 \\ \lambda_{1,a_1} & \lambda_{2,a_2} & \cdots & \lambda_{k,a_k} \\ \vdots & \vdots & \vdots & \vdots \\ \lambda_{1,a_1}^{r-1} & \lambda_{2,a_2}^{r-1} & \cdots & \lambda_{k,a_k}^{r-1} \end{bmatrix} \begin{bmatrix} c_{1,a} \\ c_{2,a} \\ \vdots \\ c_{k,a} \end{bmatrix}$$

- To encode, invert l matrices of dimension $r \times r$
- To update $c_{i,a}$, re-compute $c_{k+1,a}, \ldots, c_{k+r,a}$. To update one element of an information node, it is necessary to update at least one element in every parity node (TAMO ET AL., '14). Our codes have optimal update complexity.

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- Trace $\mathrm{tr}(x) = x + x^2 + x^{2^2} + x^{2^3}$ is a map from E to F:

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- For any $c \in E$ the values $\mathrm{tr}(c), \mathrm{tr}(\alpha c), \mathrm{tr}(\alpha^2 c), \mathrm{tr}(\alpha^3 c)$ suffice to recover c

General repair scheme

Let $F \subset E$ be finite fields, $[E : F] = l$; $\Lambda \subset E$; $|\Lambda| = \{P_1, \ldots, P_n\}$
Let $\mathcal{C} = RS(n, k, \Lambda)$ be the RS code; $r = n - k$

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For large n the repair bandwidth approaches the cut-set bound

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- We also prove a lower bound on the sub-packetization value of optimally repaired linear MDS codes

Theorem (Tamo-Ye-B., arXiv1706:00112)

- *Let* $\mathcal{C}$ *be an* $(n, k = n - r)$ *scalar linear MDS code over the field* $E = \mathbb{F}_{q^l}$
- *Let* d *be an integer satisfying* $k + 1 \leq d \leq n - 1$
- *Suppose that for any single failed node of* $\mathcal{C}$ *and any* d *helper nodes there is a linear repair scheme over* $\mathbb{F}_q$ *that uses the bandwidth* $dl/(d + 1 - k)$ *symbols of* $\mathbb{F}_q$
- *For a fixed* $s = d + 1 - k$ *and* $n, k \to \infty$ *the following bounds hold true:*

$$e^{(1+o(1))k \log k} \leq l \leq e^{(1+o(1))n \log n}.$$

References

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THANK YOU!