

MDS codes with optimal regeneration

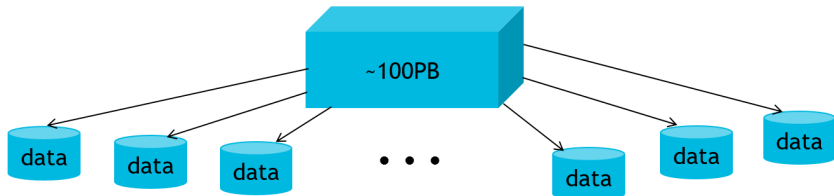
Alexander Barg

University of Maryland, College Park

June 21, 2017

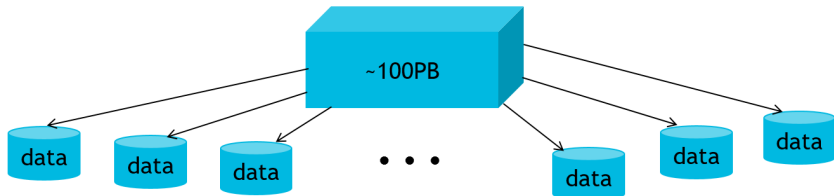


Motivation: Distributed Storage Systems (DSS)



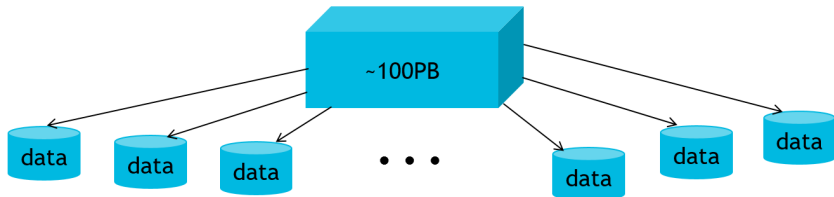
- DSS spread data across thousands of storage nodes

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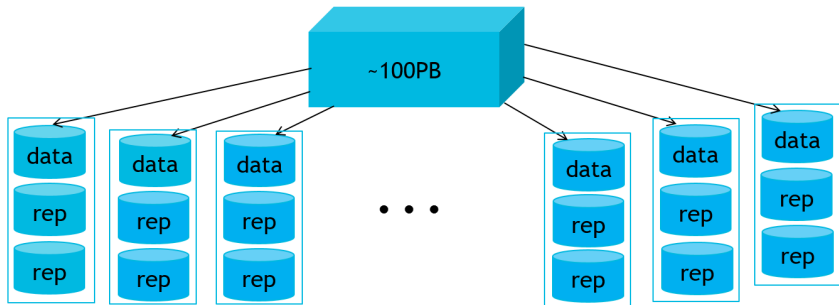
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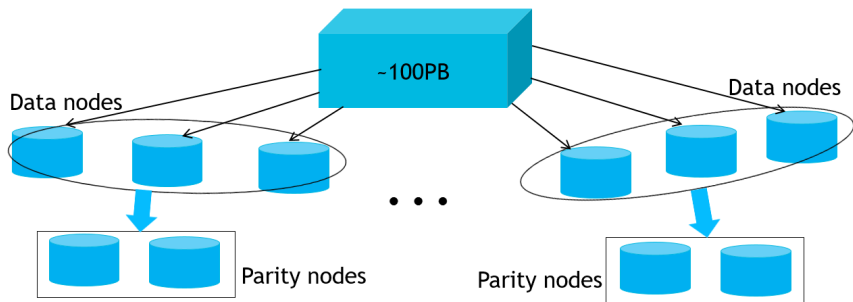
- DSS spread data across thousands of storage nodes
- Individual storage nodes fail frequently
- To protect the data we rely on erasure codes

Replication: large storage overhead



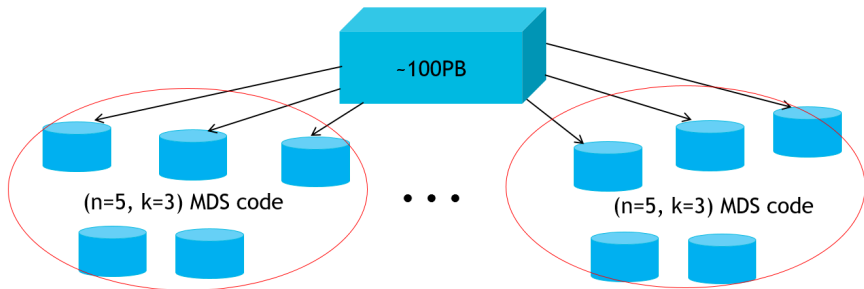
Can tolerate any 2 node failures
Storage overhead = $3\times$

MDS codes: Optimal storage efficiency



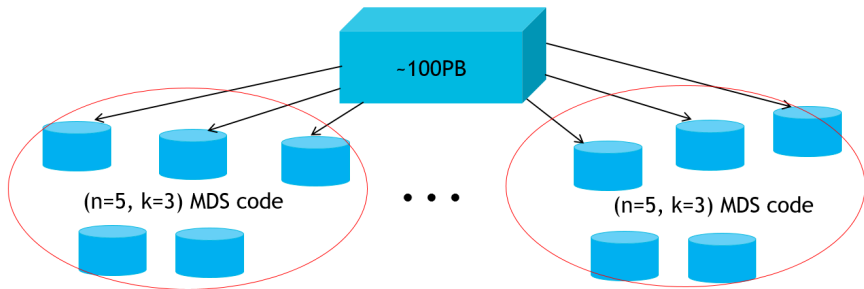
Add 2 parity nodes to every 3 data nodes
Form an $(n = 5, k = 3)$ MDS code

MDS codes: Optimal storage efficiency



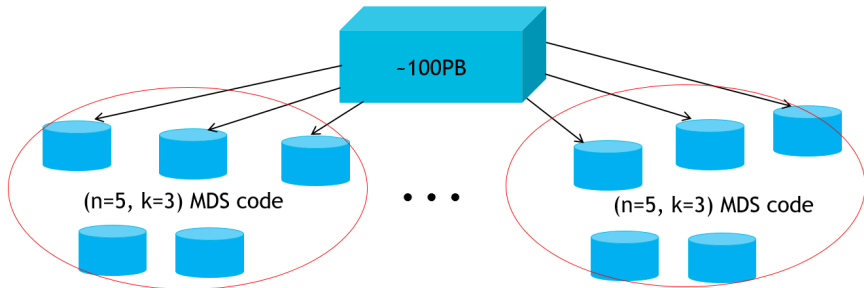
Can tolerate any 2 node failures
Storage overhead = 1.67×

MDS codes: Optimal storage efficiency



Block-based system model: Data blocks are encoded independently

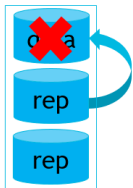
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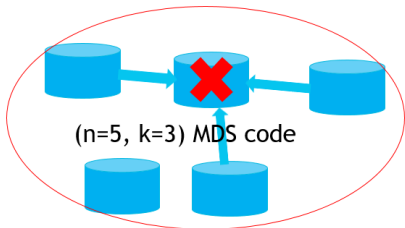
Each node stores l symbols of the block

Network flow incurred by data regeneration

Replication



Network flow = node size

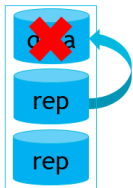


Network flow = 3 x node size

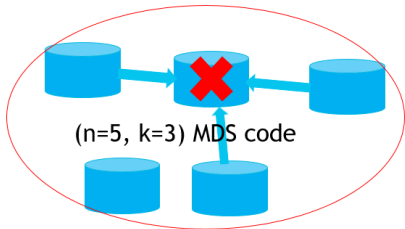
MDS code uses much more network bandwidth during data regeneration

Network flow incurred by data regeneration

Replication



Network flow = node size



Network flow = 3 x node size

MDS code uses much more network bandwidth during data regeneration

Goal: To find MDS codes using less bandwidth during data regeneration

Reed-Solomon codes

$F = \mathbb{F}_q$ the finite field of q elements

Example: $q = 8$, $\alpha^3 = \alpha + 1$

$\mathbb{F}_8 = \{0, 1, \alpha, \alpha^2, \alpha^3 = \alpha + 1, \alpha^4 = \alpha^2 + \alpha, \alpha^5 = \alpha^2 + \alpha + 1, \alpha^6 = \alpha^2 + 1\}$

The elements of \mathbb{F}_8 can also be written as binary vectors

(000), (001), (010), (101), (110), (111), (101)

Encoding example: Let $q = 8$, $(n, k) = (7, 3)$

Suppose that data symbols are $1, \alpha, \alpha$

To encode, form a polynomial $f(x) = 1 + \alpha x + \alpha x^2$ and compute the values of $f(x)$ at the points $1, \alpha, \alpha^2, \dots, \alpha^6$

x		1	α	α^2	α^3	α^4	α^5	α^6
$f(x)$		1	α^4	α^6	α^4	α	α	α^6

This encodes $k = 3$ data symbols $(1, \alpha, \alpha)$ into $n = 7$ symbols of the codeword

Reed-Solomon codes

Reed-Solomon code $\mathcal{C}(A)$ over F encodes k data symbols into n code symbols

Let $f(x) = a_0 + a_1x + \dots + a_{k-1}x^{k-1}$ be the data (represented by polynomials) and encode as follows:

$$\begin{aligned}\mathcal{C} : F^k &\rightarrow F_q^n \\ f(x) &\mapsto (f(P_1), f(P_2), \dots, f(P_n))\end{aligned}$$

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In other words, (c_1, \dots, c_n) is an RS codeword if it satisfies the following **parity-check equations**:

$$c_1 + c_2\alpha^i + c_3\alpha^{2i} \dots + c_n\alpha^{(n-1)i} = 0, \quad i = 1, \dots, n - k$$

Use of RS codes for node repair

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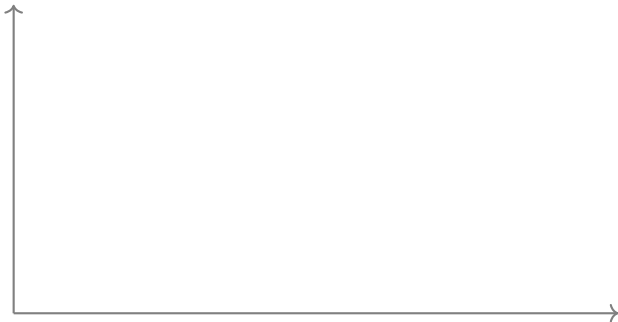
(a_0, a_1, a_2, a_3)

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$$(a_0, a_1, a_2, a_3) \rightarrow f(x) = a_0 + a_1x + a_2x^2 + a_3x^3$$

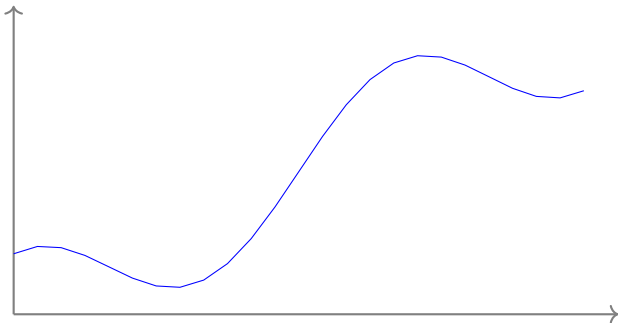
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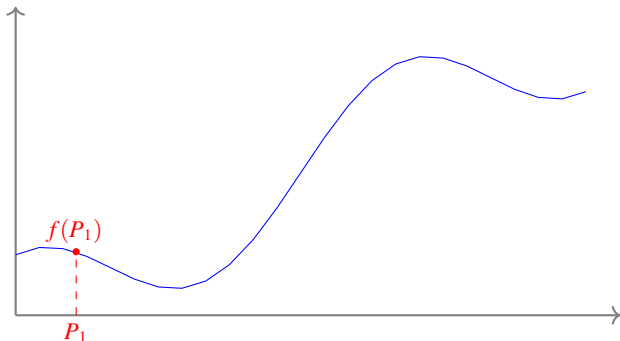
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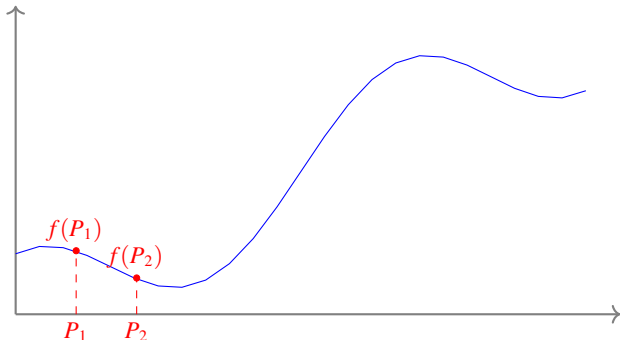
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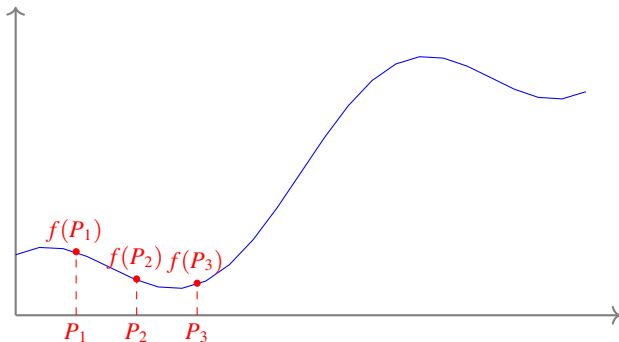
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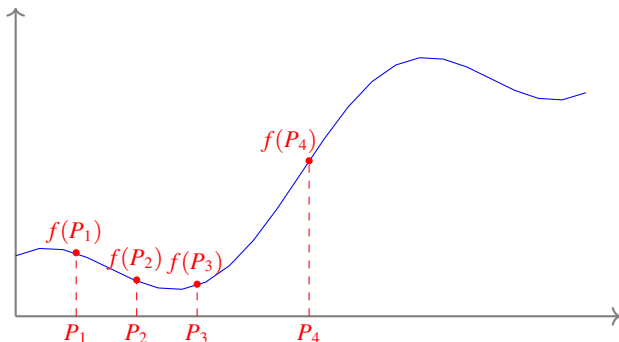
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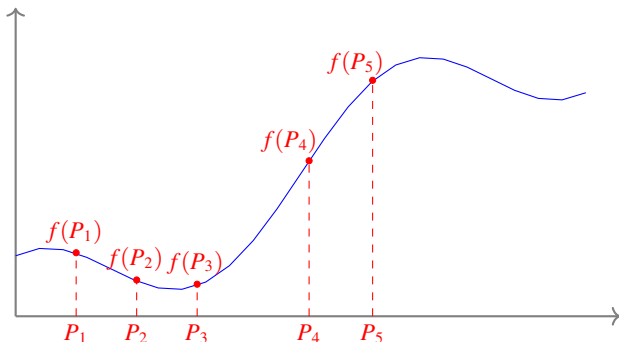
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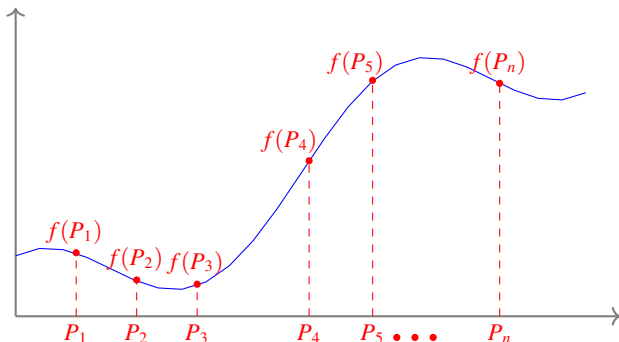
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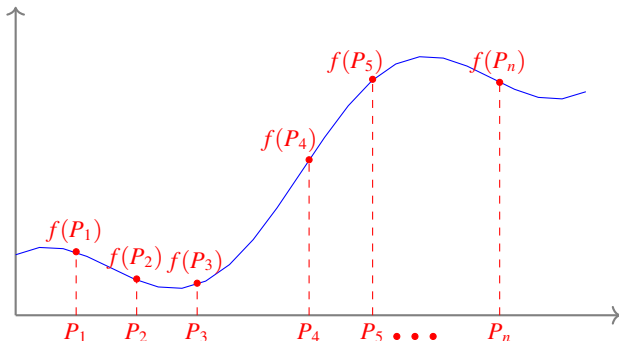
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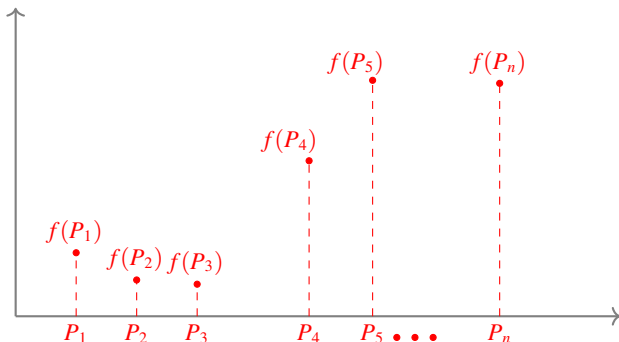
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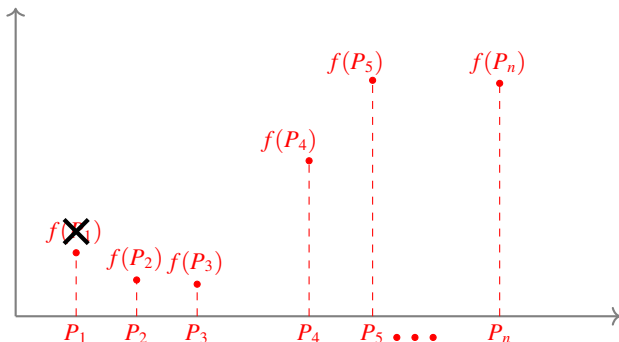
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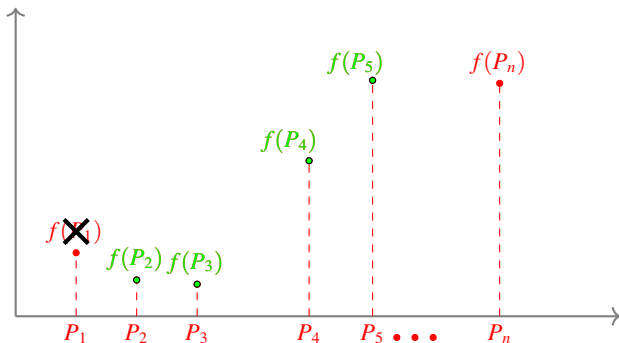
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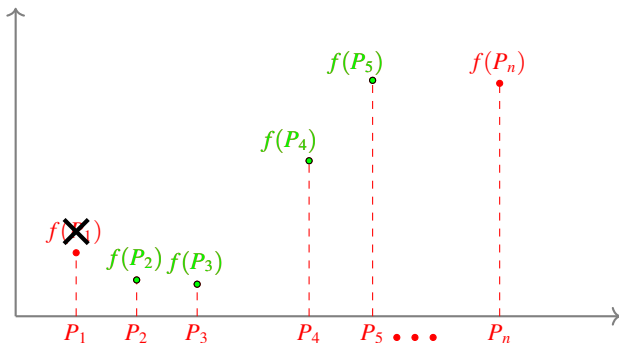
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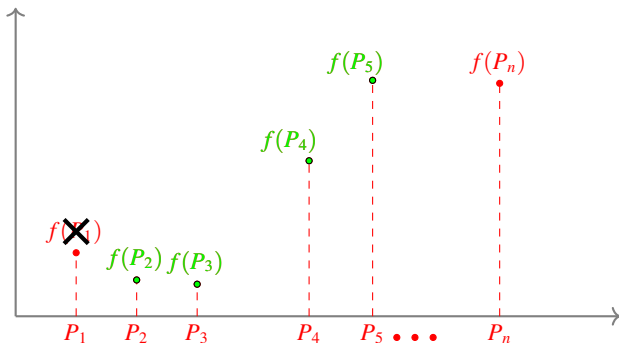
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We need to download the contents of $k = 4$ nodes to repair a failed node



MDS array codes for storage

Main idea for improvement: Sub-packetize the encoding (consider vector codes over a finite field F)

¹We use “coordinates” and “nodes” interchangeably

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Each coordinate¹ of the codeword $(C_1, C_2, \dots, C_n) \in F^n$ is an l -dimensional vector over F , so the codeword can be viewed as an $l \times n$ array over F

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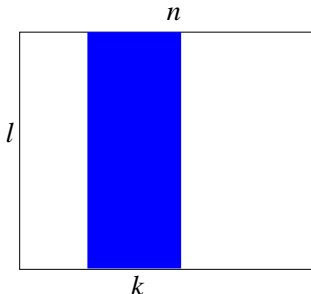


- **MDS property:** Contents of any r nodes can be determined by the other k nodes.

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MDS array codes

Consider an $\ell \times n$ array code \mathcal{C} ('vector code') over the field F



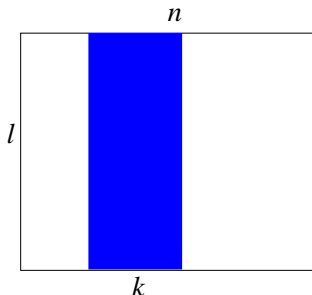
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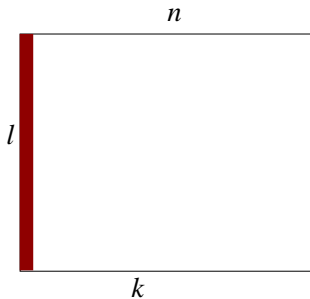
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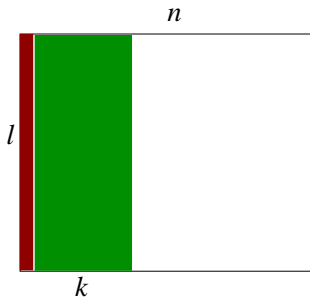
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The code \mathcal{C} is called **MDS** if any k columns suffice to recover the entire codeword
A column in this matrix is also called a **node**.

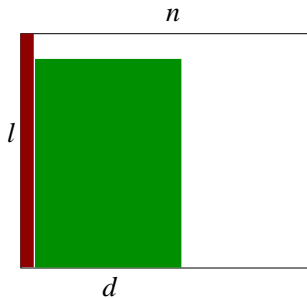
MDS array codes and repair bandwidth



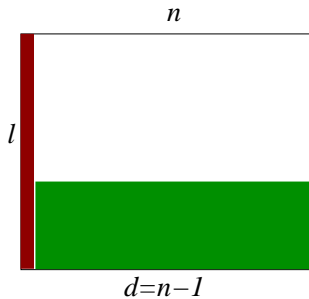
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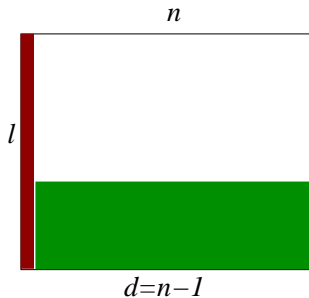
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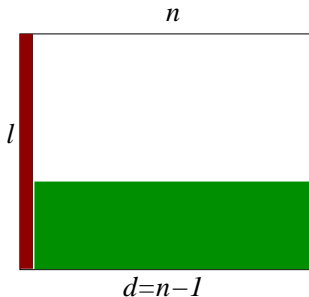


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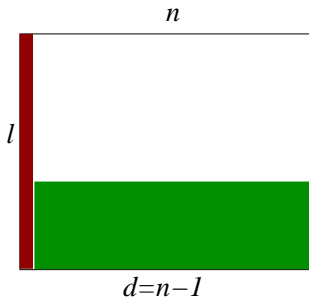
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$$\beta \geq \frac{dl}{d+1-k} \quad (\text{Dimakis et al., 2010})$$

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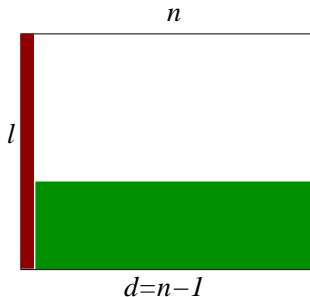


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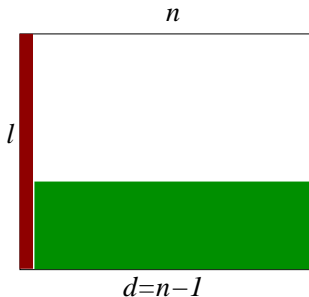
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The code meeting this bound with equality is said to have **optimal repair bandwidth**

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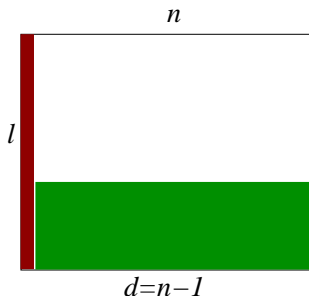
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For $d = n - 1, r = n - k$

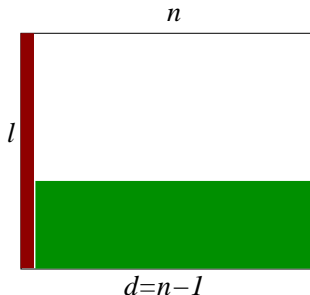
$$\beta \geq \frac{\ell}{r}(n - 1)$$

The repair problem



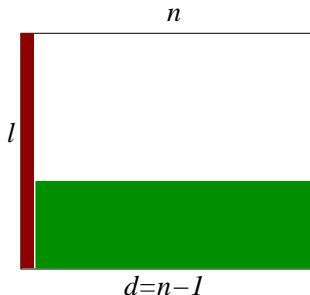
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- If the downloaded information is the same as the information accessed at the helper nodes, the code is said to support repair by transfer
- If in addition the code has optimal repair bandwidth, it is called the **optimal-access code**

Repair bandwidth of MDS codes

It is possible to generalize the bound on the minimum bandwidth to the case of $h \geq 1$ failed nodes

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The codes that meet this bound are said to have (h, d) -optimal repair property

This talk

Part I: Explicit constructions of MDS array codes with optimal repair

Part II: Asymptotically optimal family of Reed-Solomon codes

Part I: Optimal-repair MDS array codes

Previous results (a sampling):

- Low rate $k/n \leq 1/2$: Constructions of codes with $(1, n - 1)$ optimal repair (RASHMI ET AL., '11, SUH-RAMCHANDRAN, '11)
- High rate: Constructions for $r = 2, 3$ (TAMO-WANG-BRUCK '13; PAPALIOPOULOS ET AL. '13; WANG-TAMO-BRUCK '16; RAVIV-SILBERSTEIN-ETZION '17)
- High rate, any r : A number of existence results over large alphabets (CADAMBE ET AL., '12; WANG ET AL., '16; GOPARAJU-FAZELI-VARDY, '16)
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- SASIDHARAN-VAJHA-KUMAR, '16-'17

Part I: Optimal-repair MDS array codes

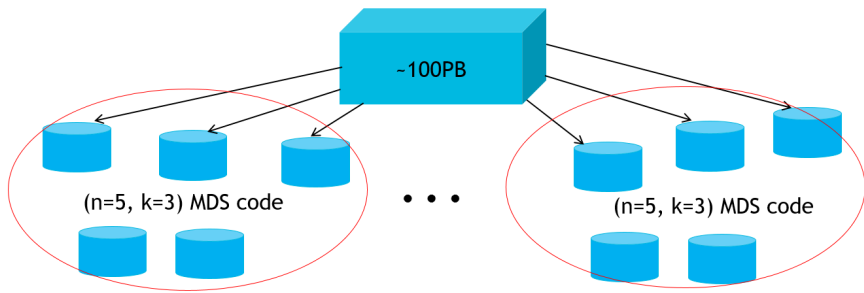
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- SASIDHARAN-VAJHA-KUMAR, '16-'17

In this talk (MIN YE and A.B., '16-'17):

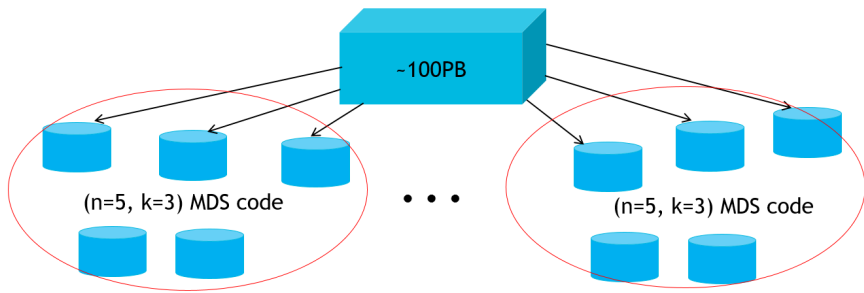
- A construction of (h, d) -optimal repair codes for any r , any $h \leq r$, and $d, k \leq d \leq n - h$
- Code families that work for all h and all d simultaneously
- Error-resilient optimal-repair codes
- Optimal-access codes with the above properties

Placement of MDS codewords in storage nodes



Any k nodes suffice to recover the data

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Each node stores l symbols of the block

General encoding method

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- $r \times n$ parity check matrix

$$\begin{bmatrix} A_{1,1} & A_{1,2} & A_{1,3} & \dots & A_{1,n} \\ A_{2,1} & A_{2,2} & A_{2,3} & \dots & A_{2,n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ A_{r,1} & A_{r,2} & A_{r,3} & \dots & A_{r,n} \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ \vdots \\ C_n \end{bmatrix} = 0$$

where each A_{ij} is an $l \times l$ matrix and C_i is an l -vector over F

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- $A_i - A_j$ invertible
- Natural choice: **Diagonal matrices**

Construction I: Optimal $(1, n - 1)$ repair MDS codes

- Take $l = r^n$; take the $l \times l$ matrix $A_i, i = 1, \dots, n$ in the form

$$A_i = \sum_{a=0}^{l-1} \lambda_{i,a_i} e_a e_a^T$$

where λ_{ij} are distinct elements of F and $(a_n, a_{n-1}, \dots, a_1)$ is r -ary expansion of a

- 1) Note that we need $r \cdot n$ distinct elements of the finite field F , i.e., $|F| \geq rn$.
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- The codeword has the form $C = (C_1, \dots, C_n)$, where $C_i = (c_{i,0}, c_{i,1}, \dots, c_{i,l-1})^T$

$$\begin{array}{cccc} c_{1,0} & c_{2,0} & \dots & c_{n,0} \\ c_{1,1} & c_{2,1} & \dots & c_{n,1} \\ \vdots & \vdots & \vdots & \vdots \\ c_{1,l-1} & c_{2,l-1} & \dots & c_{n,l-1} \end{array}$$

- Idea:** Every row forms an RS code with different evaluation points $\{P_{i,j}\}$
For $a = 0, 1, \dots, l - 1$, write r -ary expansion $a = (a_1, a_2, \dots, a_n)$
Evaluation points for a -th row: $(\lambda_{1,a_1}, \lambda_{2,a_2}, \dots, \lambda_{n,a_n})$

$(1, n - 1)$ -optimal repair property

Idea (cont'd): Let $C_i = (c_{i,a}, a = 0, 1, \dots, l - 1)$ be the missing node.

Repair the contents by **groups of size r** that differ only in position i of the expansion

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- $a(i, u) = (a_1, a_2, \dots, a_{i-1}, u, a_{i+1}, a_{i+2}, \dots, a_n), \quad 0 \leq u \leq r - 1$

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$$\underbrace{\begin{bmatrix} 1 & 1 & \dots & 1 \\ \lambda_{i,0} & \lambda_{i,1} & \dots & \lambda_{i,r-1} \\ \vdots & \vdots & \vdots & \vdots \\ \lambda_{i,0}^{r-1} & \lambda_{i,1}^{r-1} & \dots & \lambda_{i,r-1}^{r-1} \end{bmatrix}}_{\text{Vandermonde, rank} = r} \underbrace{\begin{bmatrix} c_{i,a(i,0)} \\ c_{i,a(i,1)} \\ \vdots \\ c_{i,a(i,r-1)} \end{bmatrix}}_{\text{missing information}}$$

$$\sum_{u=0}^{r-1} c_{j,a(i,u)}$$

$$+ \sum_{j \neq i}$$

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= 0

$(1, n - 1)$ -optimal repair property

- For $u = 0, 1, \dots, r - 1$ let $a(i, u) := (a_n, \dots, a_{i+1}, u, a_{i-1}, \dots, a_1)$.
- Partition the symbols on the failed node i into r^{l-1} **groups** of size r each:

$$\{c_{i,a(i,0)}, c_{i,a(i,1)}, \dots, c_{i,a(i,r-1)}\}$$

for some $a \in \{0, 1, \dots, l - 1\}$

- Say C_i is unavailable. The elements in each **group** can be found by acquiring **one element** $\sum_{u=0}^{r-1} c_{ja(i,u)}$ from each of the $n - 1$ remaining nodes.
- **Total repair bandwidth** = $(n - 1) \times 1 \times r^{l-1} = (n - 1)(l/r)$, matching the lower bound

Construction II: Codes with $(1, d)$ -optimal repair and error correction, $k < d < n - 1$

The bound on the repair bandwidth can be extended to error correction:

$$\beta(d, t) \geq \frac{(d + 2t)l}{d + 1 - k}$$

(PAWAR ET AL., '11; RASHMI ET AL., '12)

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- Construct (n, k) MDS code \mathcal{C} over $(F)^l$, $l = s^n$, $s = d + 1 - k$
- Take $|F| \geq sn$, $\{\lambda_{ij}, i = 1, \dots, n; j = 0, \dots, s - 1\} \subset F$,
- Parity-check matrix H is a block Vandermonde matrix with

$$A_i = \sum_{a=0}^{l-1} \lambda_{i, a_i} e_a e_a^T$$

where $a = (a_n, a_{n-1}, \dots, a_1)$ is the s -ary expansion of $a = 0, 1, \dots, l - 1$.

- We use the fact that the symbols transmitted from the helper nodes form a (generalized) RS code

Construction III: Codes with (h, d) -optimal repair for all h, d simultaneously

We take $s = \text{lcm}(1, 2, \dots, r)$

Sequential repair:

Let C_1, \dots, C_h be the failed nodes, $\mathcal{R}, |\mathcal{R}| = d + 2t$ be the helper set.

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- \dots
- $C_{\mathcal{R}} \cup C_1 \cup C_2 \cup \dots \cup C_{h-1}$ to repair C_h

Further extensions

1. d -optimal repair ($n, k = n - r, l = r^{n-1}$) MDS codes with **optimal access property** and error correction, $|F| \geq n + 1$
(Idea: Use permutation matrices instead of diagonal matrices)
2. Codes for optimal repair from d_1, \dots, d_m helper nodes simultaneously, $k \leq d_1, \dots, d_m < n$

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2. Codes for optimal repair from d_1, \dots, d_m helper nodes simultaneously, $k \leq d_1, \dots, d_m < n$

The codes have small encoding, repair, and update complexity.

They also afford efficient ***systematic representation***

Systematic encoding; Updates

For $a = 0, 1, \dots, l - 1$

$$\begin{aligned}
 & \begin{bmatrix} 1 & 1 & \dots & 1 \\ \lambda_{k+1, a_{k+1}} & \lambda_{k+2, a_{k+2}} & \dots & \lambda_{k+r, a_{k+r}} \\ \vdots & \vdots & \vdots & \vdots \\ \lambda_{k+1, a_{k+1}}^{r-1} & \lambda_{k+2, a_{k+2}}^{r-1} & \dots & \lambda_{k+r, a_{k+r}}^{r-1} \end{bmatrix} \begin{bmatrix} c_{k+1, a} \\ c_{k+2, a} \\ \vdots \\ c_{k+r, a} \end{bmatrix} \\
 &= - \begin{bmatrix} 1 & 1 & \dots & 1 \\ \lambda_{1, a_1} & \lambda_{2, a_2} & \dots & \lambda_{k, a_k} \\ \vdots & \vdots & \vdots & \vdots \\ \lambda_{1, a_1}^{r-1} & \lambda_{2, a_2}^{r-1} & \dots & \lambda_{k, a_k}^{r-1} \end{bmatrix} \begin{bmatrix} c_{1, a} \\ c_{2, a} \\ \vdots \\ c_{k, a} \end{bmatrix}
 \end{aligned}$$

- To encode, invert l matrices of dimension $r \times r$
- To update $c_{i, a}$, re-compute $c_{k+1, a}, \dots, c_{k+r, a}$. To update one element of an information node, it is necessary to update at least one element in every parity node (TAMO ET AL., '14). Our codes have **optimal update complexity**.

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- Let $\alpha \in E$ be such that $\alpha^4 = \alpha + 1$, then $(1, \alpha, \alpha^2, \alpha^3)$ form a basis of E over F
- Trace $\text{tr}(x) = x + x^2 + x^{2^2} + x^{2^3}$ is a map from E to F :

$$\text{tr}(0) = 0, \text{tr}(1) = 0, \text{tr}(\alpha) = 1, \quad \text{etc.}$$

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- For any $c \in E$ the values $\text{tr}(c), \text{tr}(\alpha c), \text{tr}(\alpha^2 c), \text{tr}(\alpha^3 c)$ suffice to recover c

General repair scheme

Let $F \subset E$ be finite fields, $[E : F] = l$; $\Lambda \subset E$; $|\Lambda| = \{P_1, \dots, P_n\}$

Let $\mathcal{C} = RS(n, k, \Lambda)$ be the RS code; $r = n - k$

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For large n the repair bandwidth approaches the cut-set bound

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Theorem (Tamo-Ye-B., arXiv1706:00112)

- Let \mathcal{C} be an $(n, k = n - r)$ scalar linear MDS code over the field $E = \mathbb{F}_{q^t}$
- Let d be an integer satisfying $k + 1 \leq d \leq n - 1$
- Suppose that for any single failed node of \mathcal{C} and any d helper nodes there is a **linear repair scheme** over \mathbb{F}_q that uses the bandwidth $dl/(d + 1 - k)$ symbols of \mathbb{F}_q
- For a fixed $s = d + 1 - k$ and $n, k \rightarrow \infty$ the following bounds hold true:

$$e^{(1+o(1))k \log k} \leq l \leq e^{(1+o(1))n \log n}.$$

References

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THANK YOU!