What is private information retrieval?

$x = (x_1, x_2, \ldots, x_n)$

Alice wishes to retrieve a data item $x_i$ from the database $(x_1, x_2, \ldots, x_n)$ without revealing any information about $i$ to the server.

Formal privacy condition: The distribution of randomized queries sent by the user to the server does not depend on $i$.

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**Queries**

$x_i = ?$

**Answers**

**Alice**

---

**Private information retrieval (PIR)**

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**Naive Solution:** Ask the server to send the entire database!

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**Solution:** Ask the server to send the entire database!

This is the only solution possible! Communication cost $= \Omega(n)$.

Two general classes of solutions

- **Computational PIR**
  
The server is computationally bounded + **standard cryptographic assumptions** (one-way functions, quadratic residuosity).


- **Information-theoretic PIR**
  
The database is replicated among $k \geq 2$ non-communicating servers, with guarantees of information-theoretic privacy.

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This talk: **We consider only information-theoretic PIR!**
Information-theoretic PIR: Example

Replication among $k = 4$ servers $S_1, S_2, S_3, S_4$ with communication cost of $8\sqrt{n} + 4$ bits. The database is represented as a square of side $\sqrt{n}$.

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**Query generation:**
Alice wishes to retrieve \( x_{s,t} \). She generates the vectors \( y, z \in \{0, 1\}^{\sqrt{n}} \) uniformly at random, and sends

\[
S_1 \leftarrow (y, z), \quad S_2 \leftarrow (y + e_s, z), \quad S_3 \leftarrow (y, z + e_t), \quad S_4 \leftarrow (y + e_s, z + e_t)
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**Answer computation:**
Given a query \((u, v)\), each server \( S_i \) returns the following:
\[
a = \sum_{i \in \text{supp}(u)} \sum_{j \in \text{supp}(v)} x_{i,j}
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**Reconstruction:**

![Reconstruction Diagram]

The bit \( x_{s, t} \) contributes to exactly one of the answers \( a_1, a_2, a_3, a_4 \). All other bits in the database contribute an even number of times.

It follows from 1 and 2 that:
\[ a_1 + a_2 + a_3 + a_4 = x_{s, t} \]
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\text{storage overhead} \overset{\text{def}}{=} \frac{\text{total number of bits stored on all the servers}}{\text{number of bits in the database}}
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The storage overhead of replicating the database \(k\) times is trivially \(k\). The Dvir and Gopi paper is considered a breakthrough in part because it reduces the storage overhead from \(k \geq 3\) to \(k = 2\), for the same complexity.
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This talk: The main theme

This is cryptography, people! We do the impossible for breakfast.

Open Problem: Can we achieve information-theoretic PIR with low communication cost but without doubling (or worse if $k \geq 3$) the number of bits we need to store?
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**Taking cue from distributed storage:** In practice, the database may need to be stored in a distributed manner (e.g., for security or reliability purposes).

**Key idea: Partitioning the database**

Partition the database string \( x \) into parts \( x_1, x_2, \ldots, x_s \). We will use \( m \geq k \) non-communicating servers. But **each server will store only part of the database**, so that the total number of bits stored is \( (1 + \varepsilon)n \).
Conventional $k$-server PIR

**Definition: $k$-server PIR scheme**

A $k$-server PIR scheme consists of the following: a binary string $x$ of length $n$, called the database, $k$ non-communicating servers $S_1, S_2, \ldots, S_k$ each storing a replica of $x$, a user Alice who wishes to retrieve $x_i$ for some $i \in [n]$, without revealing $i$ to any of the servers, and a $k$-server PIR protocol.

**Definition: $k$-server PIR protocol [CKGS95]**

The $k$-server PIR protocol $\mathcal{P}$ involves a triple of algorithms $Q, A, C$ and consists of the following steps:

1. Alice flips coins and uses the random outcome to invoke the query algorithm $Q(k, n; i)$ that generates a $k$-tuple of queries $q_1, q_2, \ldots, q_k$.
2. For all $j \in [k]$, Alice sends the query $q_j$ to the $j$-th server $S_j$.
3. For all $j \in [k]$, the server $S_j$ invokes the answer algorithm $A$ to respond with the answer $a_j = A(k, j; x, q_j)$.
4. Alice computes $x_i$ using the reconstruction algorithm $C(k, n; i, a_1, \ldots, a_k)$.

The three algorithms together satisfy the correctness $(C(k, n; i, a_1, \ldots, a_k) = x_i)$ and the privacy (distribution of $q_j$ independent of $i$) conditions defined earlier.
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The three algorithms together satisfy the correctness ($C(k, n; i, a_1, \ldots, a_k) = x_i$) and the privacy (distribution of $q_j$ independent of $i$) conditions defined earlier.
Conventional $k$-server PIR: Linearity

Our construction of distributed PIR schemes with low storage overhead uses two main ingredients:

1. A binary linear code $C$ with a certain special property, to be defined shortly.
2. An existing $k$-server PIR protocol in which the answer algorithm is linear in the database.

Good news: All known PIR protocols are linear!

Note: We also assume that the answer algorithm $A$ is public knowledge. This means that any server can simulate the answers of any other server.
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**Definition: Linear $k$-server PIR protocol**

A $k$-server PIR protocol $\mathcal{P}(Q, A, C)$ is **linear** if for all $x_1, x_2 \in \{0,1\}^n$ and for all possible queries $q$, the following holds:

$$A(k, j; x_1 + x_2, q) = A(k, j; x_1, q) + A(k, j; x_2, q) \quad \text{for all } j \in [k]$$
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Consider any existing 3-server PIR protocol $\mathcal{P}(Q, A, C)$, and assume it is linear. We will reduce its storage overhead from $k = 3$ to $m/s = 2$. 

We partition the database $x$ of length $n$ into 4 parts $x_1, x_2, x_3, x_4$, each of length $n/4$. These parts are distributed among 8 servers as follows:

- $S_1$: $c_1 = x_1$
- $S_5$: $c_5 = x_1 + x_2$
- $S_2$: $c_2 = x_2$
- $S_6$: $c_6 = x_2 + x_3$
- $S_3$: $c_3 = x_3$
- $S_7$: $c_7 = x_3 + x_4$
- $S_4$: $c_4 = x_4$
- $S_8$: $c_8 = x_4 + x_1$

The result is a coded PIR scheme with $s = 4$ parts $x_1, x_2, x_3, x_4$ and $m = 8$ coded shares $c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8$. 

Storage overhead $= n/s$ bits stored on $m$ servers $n$ bits in the database $= m/s = 2$. 

**Example: Coded 3-server PIR**
Example: Coded 3-server PIR

**Example: Reducing the storage overhead of 3-server PIR**

Consider *any* existing 3-server PIR protocol $\mathcal{P}(Q, A, C)$, and assume it is linear. We will reduce its storage overhead from $k = 3$ to $m/s = 2$.

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Example: How to retrieve $x_i$?

Assume, for now, that Alice wishes to read the *i*-th bit from the first part $x_1$. That is, she wants the bit $x_{1,i}$ for some $i \in [n/4]$. She proceeds as follows:

1. Alice flips coins and invokes the *query algorithm of* $\mathcal{P}(Q,A,C)$ to generate three queries $q_1, q_2, q_3 := Q(3,n/4;i)$.

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   $$(S_1, S_2, S_3, S_4, S_5, S_6, S_7, S_8) \leftarrow (q_1, q_2, q_3, q_3, q_2, q_2, q_3, q_3)$$

3. Alice ignores the answers from $S_3, S_6, S_7$ but collects the other five answers as follows:

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$$ a'_3 = a_4 + a_8 = \mathcal{A}(3,3;x_4,q_3) + \mathcal{A}(3,3;x_4+x_1,q_3) = \mathcal{A}(3,3;x_1,q_3) $$
Example: How to retrieve $x_i$?

Assume, for now, that Alice wishes to read the $i$-th bit from the first part $x_1$. That is, she wants the bit $x_{1,i}$ for some $i \in [n/4]$. She proceeds as follows:

Alice ignores the answers from $S_3, S_6, S_7$ but collects the other five answers as follows:

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Since the answer algorithm of $P(Q,A,C)$ is linear in the database, Alice can compute:

$$a'_2 = a_2 + a_5 = A(3,2;x_2,q_2) + A(3,2;x_1 + x_2,q_2) = A(3,2;x_1,q_2)$$

$$a'_3 = a_4 + a_8 = A(3,3;x_4,q_3) + A(3,3;x_4 + x_1,q_3) = A(3,3;x_1,q_3)$$

Using the reconstruction algorithm of $P(Q,A,C)$, Alice now computes $C(3,n/4;i,a_1,a'_2,a'_3)$, which is given by:

$$C(3,n/4;i,A(3,1;x_1,q_1),A(3,2;x_1,q_2),A(3,3;x_1,q_3)) = x_{1,i}$$
Example: How to retrieve $x_i$?

Now assume that Alice wishes to read the $i$-th bit from the second part $x_2$. That is, she wants the bit $x_{2,i}$ for some $i \in [n/4]$. She proceeds as follows:

1. Alice flips coins and invokes the *query algorithm of* $P(Q, A, C)$ to generate three queries $q_1, q_2, q_3 := Q(3, n/4; i)$, exactly as before.

2. She sends queries to the 8 servers as follows:

   $$(S_1, S_2, S_3, S_4, S_5, S_6, S_7, S_8) \leftarrow (q_2, q_1, q_3, q_3, q_2, q_3, q_3, q_3)$$

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5. Using the reconstruction algorithm of $P(Q, A, C)$, Alice now computes $C(3, n/4; i, a_2, a'_2, a'_3)$, which is given by:

$$C(3, n/4; i, A(3,1; x_2, q_1), A(3,2; x_2, q_2), A(3,3; x_2, q_3)) = x_{2,i}$$
Coded $k$-server PIR: Definition

**Definition: Coded $k$-server PIR scheme**

A **coded $k$-server PIR scheme with $s$ parts and $m$ shares** consists of the following ingredients:

- A binary string $x$ of length $n$, called the database, that is partitioned into $s$ parts $x_1, x_2, \ldots, x_s$, each of length $n/s$.

- Coded shares $c_1, c_2, \ldots, c_m$ of length $n/s$, where $c_j$ is a linear function of $x_1, x_2, \ldots, x_s$ for all $j \in [m]$, stored in $m$ non-communicating servers $S_1, S_2, \ldots, S_m$.

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**Theorem 1: Storage overhead of coded PIR**

The storage overhead of a coded $k$-server PIR scheme with $s$ parts and $m$ coded shares is $m/s$. 
So far, we have seen a general definition, and a single example of a coded PIR scheme with 4 parts and 8 shares that conforms to this definition.
General coded PIR schemes?

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- Why does the bit retrieval in the example work? Why does everything nicely cancel out?

- For which values of $m$, $s$, and $k$ do coded $k$-server PIR schemes with $s$ parts and $m$ shares exist?

- What about their communication complexity?

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To answer these questions, let us begin by revisiting the example.
The example revisited

In the encoding equations (★) of the example, the 8 coded shares are computed from the four database parts $x_1, x_2, x_3, x_4$ as follows:

\[
\begin{align*}
  c_1 &= x_1, & c_3 &= x_3, & c_5 &= x_1 + x_2, & c_7 &= x_3 + x_4 \\
  c_2 &= x_2, & c_4 &= x_4, & c_6 &= x_2 + x_3, & c_8 &= x_4 + x_1
\end{align*}
\]

Rewrite these equations in matrix form:

\[
\begin{pmatrix}
  c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8
\end{pmatrix} =
\begin{pmatrix}
  1 & 0 & 0 & 1 & 0 & 0 & 1 \\
  0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\
  0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\
  0 & 0 & 0 & 1 & 0 & 0 & 1 & 1
\end{pmatrix}
\begin{pmatrix}
  x_1, x_2, x_3, x_4
\end{pmatrix}
\]

Observe that each part $x_1, x_2, x_3, x_4$ of the database can be recovered from the coded shares in $k = 3$ different ways. Explicitly:

\[
\begin{align*}
  x_1 &= c_1 &= c_5 + c_2 &= c_8 + c_4 \\
  x_2 &= c_2 &= c_5 + c_1 &= c_6 + c_3 \\
  x_3 &= c_3 &= c_6 + c_2 &= c_7 + c_4 \\
  x_4 &= c_4 &= c_7 + c_3 &= c_8 + c_1
\end{align*}
\]

Moreover, each coded share $c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8$ appears in each of the four recovery equations above no more than once.
The example revisited

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Observe that each part \( x_1, x_2, x_3, x_4 \) of the database can be recovered from the coded shares in \( k = 3 \) different ways. Explicitly:

- \( x_1 = c_1 = c_5 + c_2 = c_8 + c_4 \)
- \( x_2 = c_2 = c_5 + c_1 = c_6 + c_3 \)
- \( x_3 = c_3 = c_6 + c_2 = c_7 + c_4 \)
- \( x_4 = c_4 = c_7 + c_3 = c_8 + c_1 \)

Moreover, each coded share \( c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8 \) appears in each of the four recovery equations above no more than once.
The example revisited

In the encoding equations (⋆) of the example, the 8 coded shares are computed from the four database parts \(x_1, x_2, x_3, x_4\) as follows:

\[
\begin{align*}
c_1 &= x_1, & c_3 &= x_3, & c_5 &= x_1 + x_2, & c_7 &= x_3 + x_4 \\
c_2 &= x_2, & c_4 &= x_4, & c_6 &= x_2 + x_3, & c_8 &= x_4 + x_1
\end{align*}
\]

Rewrite these equations in matrix form:

\[
\begin{bmatrix}
c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ c_6 \\ c_7 \\ c_8
\end{bmatrix} = \begin{bmatrix}1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1
\end{bmatrix} \begin{bmatrix}x_1 \\ x_2 \\ x_3 \\ x_4
\end{bmatrix}
\]

Observe that each part \(x_1, x_2, x_3, x_4\) of the database can be recovered from the coded shares in \(k = 3\) different ways. Explicitly:

\[
\begin{align*}
x_1 &= c_1 = c_5 + c_2 = c_8 + c_4 \\
x_2 &= c_2 = c_5 + c_1 = c_6 + c_3 \\
x_3 &= c_3 = c_6 + c_2 = c_7 + c_4 \\
x_4 &= c_4 = c_7 + c_3 = c_8 + c_1
\end{align*}
\]

Moreover, each coded share \(c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8\) appears in each of the four recovery equations above no more than once.
**The example revisited**

In the encoding equations (*) of the example, the 8 coded shares are computed from the four database parts $x_1, x_2, x_3, x_4$ as follows:

\[
\begin{align*}
c_1 &= x_1, & c_3 &= x_3, & c_5 &= x_1 + x_2, & c_7 &= x_3 + x_4 \\
c_2 &= x_2, & c_4 &= x_4, & c_6 &= x_2 + x_3, & c_8 &= x_4 + x_1
\end{align*}
\]

Rewrite these equations in matrix form:

\[
\begin{pmatrix} c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8 \end{pmatrix} = \begin{pmatrix} x_1, x_2, x_3, x_4 \end{pmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \end{bmatrix}
\]

Observe that each part $x_1, x_2, x_3, x_4$ of the database can be recovered from the coded shares in $k = 3$ different ways. Explicitly:

\[
\begin{align*}
x_1 &= c_1 = c_5 + c_2 = c_8 + c_4 \\
x_2 &= c_2 = c_5 + c_1 = c_6 + c_3 \\
x_3 &= c_3 = c_6 + c_2 = c_7 + c_4 \\
x_4 &= c_4 = c_7 + c_3 = c_8 + c_1
\end{align*}
\]

Moreover, each coded share $c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8$ appears in each of the four recovery equations above no more than once.
**Definition: k-server PIR matrix**

Let $e_i$ denote the binary unit vector with 1 in position $i$ and zeros elsewhere. An $s \times m$ binary matrix $G$ is said to have property $P_k$ if for all $i \in [s]$ there exist $k$ disjoint sets of columns of $G$ that add to $e_i$. A matrix that has property $P_k$ is also said to be a $k$-server PIR matrix.
PIR matrix and PIR codes

Definition: \( k \)-server PIR matrix

Let \( e_i \) denote the binary unit vector with 1 in position \( i \) and zeros elsewhere. An \( s \times m \) binary matrix \( G \) is said to have property \( P_k \) if for all \( i \in [s] \) there exist \( k \) disjoint sets of columns of \( G \) that add to \( e_i \). A matrix that has property \( P_k \) is also said to be a \( k \)-server PIR matrix.

Example: \( 4 \times 8 \) matrix with property \( P_3 \)

\[
\begin{bmatrix}
1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 1
\end{bmatrix}
\]

Note: This is the encoding matrix for the PIR scheme in our example.
Definition: *k*-server PIR matrix

Let $e_i$ denote the binary unit vector with 1 in position $i$ and zeros elsewhere. An $s \times m$ binary matrix $G$ is said to have property $P_k$ if for all $i \in [s]$ there exist $k$ disjoint sets of columns of $G$ that add to $e_i$. A matrix that has property $P_k$ is also said to be a *k*-server PIR matrix.

Example: $4 \times 8$ matrix with property $P_3$

\[
\begin{bmatrix}
1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\
\end{bmatrix}
\]

$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

Note: This is the encoding matrix for the PIR scheme in our example.
Definition: $k$-server PIR matrix

Let $e_i$ denote the binary unit vector with 1 in position $i$ and zeros elsewhere. An $s \times m$ binary matrix $G$ is said to have property $P_k$ if for all $i \in [s]$ there exist $k$ disjoint sets of columns of $G$ that add to $e_i$. A matrix that has property $P_k$ is also said to be a $k$-server PIR matrix.

Example: $4 \times 8$ matrix with property $P_3$

\[
\begin{bmatrix}
1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\
\end{bmatrix}
\]

$e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

Note: This is the encoding matrix for the PIR scheme in our example.
**PIR matrix and PIR codes**

**Definition: k-server PIR matrix**
Let $e_i$ denote the binary unit vector with 1 in position $i$ and zeros elsewhere. An $s \times m$ binary matrix $G$ is said to have property $P_k$ if for all $i \in [s]$ there exist $k$ disjoint sets of columns of $G$ that add to $e_i$. A matrix that has property $P_k$ is also said to be a $k$-server PIR matrix.

**Example: 4 × 8 matrix with property $P_3$**

\[
\begin{bmatrix}
1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\
\end{bmatrix}
\]

$e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$

**Note:** This is the encoding matrix for the PIR scheme in our example.
Definition: $k$-server PIR matrix

Let $e_i$ denote the binary unit vector with 1 in position $i$ and zeros elsewhere. An $s \times m$ binary matrix $G$ is said to have property $P_k$ if for all $i \in [s]$ there exist $k$ disjoint sets of columns of $G$ that add to $e_i$. A matrix that has property $P_k$ is also said to be a $k$-server PIR matrix.

Example: $4 \times 8$ matrix with property $P_3$

\[
\begin{bmatrix}
1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 1
\end{bmatrix}
\]

$e_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

Note: This is the encoding matrix for the PIR scheme in our example.
**Definition: k-server PIR matrix**

Let $e_i$ denote the binary unit vector with 1 in position $i$ and zeros elsewhere. An $s \times m$ binary matrix $G$ is said to have **property $P_k$** if for all $i \in [s]$ there exist $k$ disjoint sets of columns of $G$ that add to $e_i$. A matrix that has property $P_k$ is also said to be a $k$-server **PIR matrix**.

**Example: 4 × 8 matrix with property $P_3$**

\[
\begin{bmatrix}
1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 1
\end{bmatrix}
\]

$\iff e_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

**Note:** This is the encoding matrix for the PIR scheme in our example.

**Definition: k-server PIR code**

A binary linear code $C$ of length $m$ and dimension $s$ will be called a $k$-server **PIR code** if there exists a generator matrix $G$ for $C$ with property $P_k$. 
Recovery equations from PIR codes

Lemma 2: Disjoint recovery sets

Let $C$ be a $k$-server PIR code and let $G$ be an $s \times m$ generator matrix for $C$ with property $P_k$. Let $c = xG$ be the encoding of a message $x = (x_1, x_2, \ldots, x_s)$. Then for all $i \in [s]$, there exist $k$ disjoint recovery sets $R_1, R_2, \ldots, R_k$ such that

$$x_i = \sum_{j \in R_1} c_j = \sum_{j \in R_2} c_j = \cdots = \sum_{j \in R_k} c_j$$

**Proof.** Let $g_1, g_2, \ldots, g_m$ denote the columns of $G$. Then $c = xG$ can be written in terms of the inner products of these columns with $x$, as follows:

$$c = (c_1, c_2, \ldots, c_m) = (\langle x, g_1 \rangle, \langle x, g_2 \rangle, \ldots, \langle x, g_m \rangle)$$

Now suppose that for some set of indices $R = \{j_1, j_2, \ldots, j_r\} \subseteq [m]$, the corresponding columns of $G$ add to the unit vector $e_i$. Then

$$c_{j_1} + \cdots + c_{j_r} = \langle x, g_{j_1} \rangle + \cdots + \langle x, g_{j_r} \rangle = \langle x, g_{j_1} + \cdots + g_{j_r} \rangle = \langle x, e_i \rangle = x_i$$

It follows from the above that the recovery sets $R_1, R_2, \ldots, R_k \subseteq [m]$, are simply the indices of the disjoint sets of columns of $G$ that add up to $e_i$. □
Recovery equations from PIR codes

Lemma 2: Disjoint recovery sets

Let $C$ be a $k$-server PIR code and let $G$ be an $s \times m$ generator matrix for $C$ with property $P_k$. Let $c = xG$ be the encoding of a message $x = (x_1, x_2, \ldots, x_s)$. Then for all $i \in [s]$, there exist $k$ disjoint recovery sets $R_1, R_2, \ldots, R_k$ such that

$$x_i = \sum_{j \in R_1} c_j = \sum_{j \in R_2} c_j = \cdots = \sum_{j \in R_k} c_j$$

Proof. Let $g_1, g_2, \ldots, g_m$ denote the columns of $G$. Then $c = xG$ can be written in terms of the inner products of these columns with $x$, as follows:

$$c = (c_1, c_2, \ldots, c_m) = (\langle x, g_1 \rangle, \langle x, g_2 \rangle, \ldots, \langle x, g_m \rangle)$$

Now suppose that for some set of indices $R = \{j_1, j_2, \ldots, j_r\} \subseteq [m]$, the corresponding columns of $G$ add to the unit vector $e_i$. Then

$$c_{j_1} + \cdots + c_{j_r} = \langle x, g_{j_1} \rangle + \cdots + \langle x, g_{j_r} \rangle = \langle x, g_{j_1} + \cdots + g_{j_r} \rangle = \langle x, e_i \rangle = x_i$$

It follows from the above that the recovery sets $R_1, R_2, \ldots, R_k \subseteq [m]$, are simply the indices of the disjoint sets of columns of $G$ that add up to $e_i$. □
Recovery equations from PIR codes

Lemma 2: Disjoint recovery sets

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$$x_i = \sum_{j \in R_1} c_j = \sum_{j \in R_2} c_j = \cdots = \sum_{j \in R_k} c_j$$

**Proof.** Let $g_1, g_2, \ldots, g_m$ denote the columns of $G$. Then $c = xG$ can be written in terms of the inner products of these columns with $x$, as follows:

$$c = (c_1, c_2, \ldots, c_m) = (\langle x, g_1 \rangle, \langle x, g_2 \rangle, \ldots, \langle x, g_m \rangle)$$

Now suppose that for some set of indices $R = \{j_1, j_2, \ldots, j_r\} \subseteq [m]$, the corresponding columns of $G$ add to the unit vector $e_i$. Then

$$c_{j_1} + \cdots + c_{j_r} = \langle x, g_{j_1} \rangle + \cdots + \langle x, g_{j_r} \rangle = \langle x, g_{j_1} + \cdots + g_{j_r} \rangle = \langle x, e_i \rangle = x_i$$

It follows from the above that the recovery sets $R_1, R_2, \ldots, R_k \subseteq [m]$, are simply the indices of the disjoint sets of columns of $G$ that add up to $e_i$. □
Lemma 2: Disjoint recovery sets

Let $C$ be a $k$-server PIR code and let $G$ be an $s \times m$ generator matrix for $C$ with property $P_k$. Let $c = xG$ be the encoding of a message $x = (x_1, x_2, \ldots, x_s)$. Then for all $i \in [s]$, there exist $k$ disjoint recovery sets $\mathcal{R}_1, \mathcal{R}_2, \ldots, \mathcal{R}_k$ such that

$$x_i = \sum_{j \in \mathcal{R}_1} c_j = \sum_{j \in \mathcal{R}_2} c_j = \cdots = \sum_{j \in \mathcal{R}_k} c_j$$

**Proof.** Let $g_1, g_2, \ldots, g_m$ denote the columns of $G$. Then $c = xG$ can be written in terms of the inner products of these columns with $x$, as follows:

$$c = (c_1, c_2, \ldots, c_m) = (\langle x, g_1 \rangle, \langle x, g_2 \rangle, \ldots, \langle x, g_m \rangle)$$

Now suppose that for some set of indices $\mathcal{R} = \{j_1, j_2, \ldots, j_r\} \subseteq [m]$, the corresponding columns of $G$ add to the unit vector $e_i$. Then

$$c_{j_1} + \cdots + c_{j_r} = \langle x, g_{j_1} \rangle + \cdots + \langle x, g_{j_r} \rangle = \langle x, g_{j_1} + \cdots + g_{j_r} \rangle = \langle x, e_i \rangle = x_i$$

It follows from the above that the recovery sets $\mathcal{R}_1, \mathcal{R}_2, \ldots, \mathcal{R}_k \subseteq [m]$, are simply the indices of the disjoint sets of columns of $G$ that add up to $e_i$. \qed
Lemma 2: Disjoint recovery sets

Let \( C \) be a \( k \)-server PIR code and let \( G \) be an \( s \times m \) generator matrix for \( C \) with property \( P_k \). Let \( c = xG \) be the encoding of a message \( x = (x_1, x_2, \ldots, x_s) \). Then for all \( i \in [s] \), there exist \( k \) disjoint recovery sets \( R_1, R_2, \ldots, R_k \) such that

\[
x_i = \sum_{j \in R_1} c_j = \sum_{j \in R_2} c_j = \cdots = \sum_{j \in R_k} c_j
\]

**Proof.** Let \( g_1, g_2, \ldots, g_m \) denote the columns of \( G \). Then \( c = xG \) can be written in terms of the inner products of these columns with \( x \), as follows:

\[
c = (c_1, c_2, \ldots, c_m) = (\langle x, g_1 \rangle, \langle x, g_2 \rangle, \ldots, \langle x, g_m \rangle)
\]

Now suppose that for some set of indices \( R = \{j_1, j_2, \ldots, j_r\} \subseteq [m] \), the corresponding columns of \( G \) add to the unit vector \( e_i \). Then

\[
c_{j_1} + \cdots + c_{j_r} = \langle x, g_{j_1} \rangle + \cdots + \langle x, g_{j_r} \rangle = \langle x, g_{j_1} + \cdots + g_{j_r} \rangle = \langle x, e_i \rangle = x_i
\]

It follows from the above that the recovery sets \( R_1, R_2, \ldots, R_k \subseteq [m] \), are simply the indices of the disjoint sets of columns of \( G \) that add up to \( e_i \).
Recovery equations from PIR codes

Lemma 2: Disjoint recovery sets

Let $C$ be a $k$-server PIR code and let $G$ be an $s \times m$ generator matrix for $C$ with property $P_k$. Let $c = xG$ be the encoding of a message $x = (x_1, x_2, \ldots, x_s)$. Then for all $i \in [s]$, there exist $k$ disjoint recovery sets $R_1, R_2, \ldots, R_k$ such that

$$x_i = \sum_{j \in R_1} c_j = \sum_{j \in R_2} c_j = \cdots = \sum_{j \in R_k} c_j$$

**Proof.** Let $g_1, g_2, \ldots, g_m$ denote the columns of $G$. Then $c = xG$ can be written in terms of the inner products of these columns with $x$, as follows:

$$c = (c_1, c_2, \ldots, c_m) = (\langle x, g_1 \rangle, \langle x, g_2 \rangle, \ldots, \langle x, g_m \rangle)$$

Now suppose that for some set of indices $R = \{j_1, j_2, \ldots, j_r\} \subseteq [m]$, the corresponding columns of $G$ add to the unit vector $e_i$. Then

$$c_{j_1} + \cdots + c_{j_r} = \langle x, g_{j_1} \rangle + \cdots + \langle x, g_{j_r} \rangle = \langle x, g_{j_1} + \cdots + g_{j_r} \rangle = \langle x, e_i \rangle = x_i$$

It follows from the above that the recovery sets $R_1, R_2, \ldots, R_k \subseteq [m]$, are simply the indices of the disjoint sets of columns of $G$ that add up to $e_i$. $\square$
Recovery equations from PIR codes

**Lemma 2: Disjoint recovery sets**

Let $C$ be a $k$-server PIR code and let $G$ be an $s \times m$ generator matrix for $C$ with property $P_k$. Let $c = xG$ be the encoding of a message $x = (x_1, x_2, \ldots, x_s)$. Then for all $i \in [s]$, there exist $k$ disjoint recovery sets $R_1, R_2, \ldots, R_k$ such that

$$x_i = \sum_{j \in R_1} c_j = \sum_{j \in R_2} c_j = \cdots = \sum_{j \in R_k} c_j$$

**Proof.** Let $g_1, g_2, \ldots, g_m$ denote the columns of $G$. Then $c = xG$ can be written in terms of the inner products of these columns with $x$, as follows:

$$c = (c_1, c_2, \ldots, c_m) = (\langle x, g_1 \rangle, \langle x, g_2 \rangle, \ldots, \langle x, g_m \rangle)$$

Now suppose that for some set of indices $R = \{j_1, j_2, \ldots, j_r\} \subseteq [m]$, the corresponding columns of $G$ add to the unit vector $e_i$. Then

$$c_{j_1} + \cdots + c_{j_r} = \langle x, g_{j_1} \rangle + \cdots + \langle x, g_{j_r} \rangle = \langle x, g_{j_1} + \cdots + g_{j_r} \rangle = \langle x, e_i \rangle = x_i$$

It follows from the above that the recovery sets $R_1, R_2, \ldots, R_k \subseteq [m]$, are simply the indices of the disjoint sets of columns of $G$ that add up to $e_i$. $\square$
Construction of coded PIR schemes

Theorem 3: Coded PIR schemes from PIR codes

Suppose there exists a $k$-server PIR code $C$ of length $m$ and dimension $s$ and a $k$-server linear PIR protocol $P(Q, A, C)$. Then there exists a coded PIR scheme with $s$ parts and $m$ shares along with the corresponding coded PIR protocol.

Proof. Let $G$ be a generator matrix for $C$ with property $P_k$. Then the coded shares are computed from the database parts $x_1, x_2, \ldots, x_s$ as follows:

$$(c_1, c_2, \ldots, c_m) = (x_1, x_2, \ldots, x_s) G$$

Assume Alice wishes to read the $i$-th bit from the $\ell$-th part of the database, namely the bit $x_{\ell,i}$ for some $i \in [n/s]$. She will proceed as follows.

1. Alice invokes the query algorithm of $P(Q, A, C)$ to generate $k$ randomized queries $q_1, q_2, \ldots, q_k := Q(k, n/s; i)$.

2. She next finds $k$ disjoint recovery sets $R_1, R_2, \ldots, R_k \subseteq [m]$ such that

$$x_{\ell} = \sum_{j \in R_1} c_j = \sum_{j \in R_2} c_j = \cdots = \sum_{j \in R_k} c_j$$

Such sets exist by Lemma 2. They are used to determine how to assign the queries $q_1, q_2, \ldots, q_k$ to the servers $S_1, S_2, \ldots, S_m$. 
Theorem 3: Coded PIR schemes from PIR codes

Suppose there exists a \( k \)-server PIR code \( C \) of length \( m \) and dimension \( s \) and a \( k \)-server linear PIR protocol \( P(Q,A,C) \). Then there exists a coded PIR scheme with \( s \) parts and \( m \) shares along with the corresponding coded PIR protocol.

Proof. Let \( G \) be a generator matrix for \( C \) with property \( P_k \). Then the coded shares are computed from the database parts \( x_1, x_2, \ldots, x_s \) as follows:

\[
(c_1, c_2, \ldots, c_m) = (x_1, x_2, \ldots, x_s) G
\]

Assume Alice wishes to read the \( i \)-th bit from the \( \ell \)-th part of the database, namely the bit \( x_{\ell,i} \) for some \( i \in [n/s] \). She will proceed as follows.

1. Alice invokes the query algorithm of \( P(Q,A,C) \) to generate \( k \) randomized queries \( q_1, q_2, \ldots, q_k := Q(k, n/s; i) \).
2. She next finds \( k \) disjoint recovery sets \( R_1, R_2, \ldots, R_k \subseteq [m] \) such that

\[
x_\ell = \sum_{j \in R_1} c_j = \sum_{j \in R_2} c_j = \cdots = \sum_{j \in R_k} c_j
\]

Such sets exist by Lemma 2. They are used to determine how to assign the queries \( q_1, q_2, \ldots, q_k \) to the servers \( S_1, S_2, \ldots, S_m \).
Theorem 3: Coded PIR schemes from PIR codes
Suppose there exists a \( k \)-server PIR code \( C \) of length \( m \) and dimension \( s \) and a \( k \)-server linear PIR protocol \( \mathcal{P}(Q, A, C) \). Then there exists a coded PIR scheme with \( s \) parts and \( m \) shares along with the corresponding coded PIR protocol.

**Proof.** Let \( G \) be a generator matrix for \( C \) with property \( P_k \). Then the coded shares are computed from the database parts \( x_1, x_2, \ldots, x_s \) as follows:

\[
(c_1, c_2, \ldots, c_m) = (x_1, x_2, \ldots, x_s) G
\]

Assume Alice wishes to read the \( i \)-th bit from the \( \ell \)-th part of the database, namely the bit \( x_{\ell,i} \) for some \( i \in \lfloor n/s \rfloor \). She will proceed as follows.

1. Alice invokes the query algorithm of \( \mathcal{P}(Q, A, C) \) to generate \( k \) randomized queries \( q_1, q_2, \ldots, q_k := Q(k, n/s; i) \).

2. She next finds \( k \) disjoint recovery sets \( \mathcal{R}_1, \mathcal{R}_2, \ldots, \mathcal{R}_k \subseteq [m] \) such that

\[
x_{\ell,i} = \sum_{j \in \mathcal{R}_1} c_j = \sum_{j \in \mathcal{R}_2} c_j = \cdots = \sum_{j \in \mathcal{R}_k} c_j
\]

Such sets exist by Lemma 2. They are used to determine how to assign the queries \( q_1, q_2, \ldots, q_k \) to the servers \( S_1, S_2, \ldots, S_m \).
Construction of coded PIR schemes

Theorem 3: Coded PIR schemes from PIR codes

Suppose there exists a \( k \)-server PIR code \( C \) of length \( m \) and dimension \( s \) and a \( k \)-server linear PIR protocol \( \mathcal{P}(Q,A,C) \). Then there exists a coded PIR scheme with \( s \) parts and \( m \) shares along with the corresponding coded PIR protocol.

Proof. Let \( G \) be a generator matrix for \( C \) with property \( P_k \). Then the coded shares are computed from the database parts \( x_1, x_2, \ldots, x_s \) as follows:

\[(c_1, c_2, \ldots, c_m) = (x_1, x_2, \ldots, x_s) G\]

Assume Alice wishes to read the \( i \)-th bit from the \( \ell \)-th part of the database, namely the bit \( x_{\ell,i} \) for some \( i \in [n/s] \). She will proceed as follows.

1. Alice invokes the query algorithm of \( \mathcal{P}(Q,A,C) \) to generate \( k \) randomized queries \( q_1, q_2, \ldots, q_k := Q(k, n/s; i) \).

2. She next finds \( k \) disjoint recovery sets \( \mathcal{R}_1, \mathcal{R}_2, \ldots, \mathcal{R}_k \subseteq [m] \) such that

\[x_{\ell} = \sum_{j \in \mathcal{R}_1} c_j = \sum_{j \in \mathcal{R}_2} c_j = \cdots = \sum_{j \in \mathcal{R}_k} c_j\]

Such sets exist by Lemma 2. They are used to determine how to assign the queries \( q_1, q_2, \ldots, q_k \) to the servers \( S_1, S_2, \ldots, S_m \).
Theorem 3: Coded PIR schemes from PIR codes

Suppose there exists a $k$-server PIR code $C$ of length $m$ and dimension $s$ and a $k$-server linear PIR protocol $P(Q, A, C)$. Then there exists a coded PIR scheme with $s$ parts and $m$ shares along with the corresponding coded PIR protocol.

**Proof.** Let $G$ be a generator matrix for $C$ with property $P_k$. Then the coded shares are computed from the database parts $x_1, x_2, \ldots, x_s$ as follows:

$$(c_1, c_2, \ldots, c_m) = (x_1, x_2, \ldots, x_s) G$$

Assume Alice wishes to read the $i$-th bit from the $\ell$-th part of the database, namely the bit $x_{\ell, i}$ for some $i \in [n/s]$. She will proceed as follows.

1. Alice invokes the query algorithm of $P(Q, A, C)$ to generate $k$ randomized queries $q_1, q_2, \ldots, q_k := Q(k, n/s; i)$.

2. She next finds $k$ disjoint recovery sets $\mathcal{R}_1, \mathcal{R}_2, \ldots, \mathcal{R}_k \subseteq [m]$ such that

$$x_{\ell} = \sum_{j \in \mathcal{R}_1} c_j = \sum_{j \in \mathcal{R}_2} c_j = \cdots = \sum_{j \in \mathcal{R}_k} c_j$$

Such sets exist by Lemma 2. They are used to determine how to assign the queries $q_1, q_2, \ldots, q_k$ to the servers $S_1, S_2, \ldots, S_m$. 
3 Let $R = R_1 \cup R_2 \cdots \cup R_k$ be the union of the $k$ recovery sets. For each $j \in R$, Alice finds the unique $t \in [k]$ such that $j \in R_t$ and sets $q^*_j = q_t$. For $j \notin R$, the query $q^*_j$ can be set arbitrarily (say $q^*_j = q_1$), since the response from $S_j$ will be ignored. Alice sends the queries to servers as follows:

$$(S_1, S_2, \ldots, S_m) \leftarrow (q^*_1, q^*_2, \ldots, q^*_m)$$

**Note:** The privacy of the queries $q^*_1, q^*_2, \ldots, q^*_m$ is inherited from the original PIR protocol $P(Q, A, C)$ being emulated.

4 Alice collects the answers $a_j = A(k, j; c_j, q^*_j) = A(k, t; c_j, q_t)$ from the servers, for all $j \in R$, and computes:

$$a'_t \overset{\text{def}}{=} \sum_{j \in R_t} A(k, t; c_j, q_t) = A(k, t; \sum_{j \in R_t} c_j, q_t) = A(k, t; x_\ell, q_t)$$

for $t = 1, 2, \ldots, k$, where the first equality follows from the linearity of the answer algorithm $A$ and the second from the recovery equations for $x_\ell$.

5 Alice completes the retrieval by invoking the reconstruction algorithm of the emulated protocol $P(Q, A, C)$ as follows:

$$C(k, n/s; i, a'_1, \ldots, a'_k) = C(k, n/s; i, A(k, 1; x_\ell, q_1), \ldots, A(k, k; x_\ell, q_k)) = x_{\ell, i}$$
Proof of main theorem ...continued

Let $\mathcal{R} = \mathcal{R}_1 \cup \mathcal{R}_2 \cdots \cup \mathcal{R}_k$ be the union of the $k$ recovery sets. For each $j \in \mathcal{R}$, Alice finds the unique $t \in [k]$ such that $j \in \mathcal{R}_t$ and sets $q^*_j = q_t$. For $j \notin \mathcal{R}$, the query $q^*_j$ can be set arbitrarily (say $q^*_j = q_1$), since the response from $S_j$ will be ignored. Alice sends the queries to servers as follows:

$$(S_1, S_2, \ldots, S_m) \leftarrow (q_1^*, q_2^*, \ldots, q_m^*)$$

Note: The privacy of the queries $q_1^*, q_2^*, \ldots, q_m^*$ is inherited from the original PIR protocol $\mathcal{P}(\mathcal{Q}, \mathcal{A}, \mathcal{C})$ being emulated.

Alice collects the answers $a_j = \mathcal{A}(k,j; c_j, q^*_j) = \mathcal{A}(k,t; c_j, q_t)$ from the servers, for all $j \in \mathcal{R}$, and computes:

$$a'_t \stackrel{\text{def}}{=} \sum_{j \in \mathcal{R}_t} \mathcal{A}(k,t; c_j, q_t) = \mathcal{A}(k,t; \sum_{j \in \mathcal{R}_t} c_j, q_t) = \mathcal{A}(k,t; x_\ell, q_t)$$

for $t = 1, 2, \ldots, k$, where the first equality follows from the linearity of the answer algorithm $\mathcal{A}$ and the second from the recovery equations for $x_\ell$.

Alice completes the retrieval by invoking the reconstruction algorithm of the emulated protocol $\mathcal{P}(\mathcal{Q}, \mathcal{A}, \mathcal{C})$ as follows:

$$\mathcal{C}(k,n/s;i,a_1',\ldots,a_k') = \mathcal{C}(k,n/s;i,\mathcal{A}(k,1;x_\ell,q_1),\ldots,\mathcal{A}(k,k;x_\ell,q_k)) = x_{\ell,i}$$
Let $\mathcal{R} = \mathcal{R}_1 \cup \mathcal{R}_2 \cdots \cup \mathcal{R}_k$ be the union of the $k$ recovery sets. For each $j \in \mathcal{R}$, Alice finds the unique $t \in [k]$ such that $j \in \mathcal{R}_t$ and sets $q_j^* = q_t$. For $j \not\in \mathcal{R}$, the query $q_j^*$ can be set arbitrarily (say $q_j^* = q_1$), since the response from $S_j$ will be ignored. Alice sends the queries to servers as follows:

$$(S_1, S_2, \ldots, S_m) \leftarrow (q_1^*, q_2^*, \ldots, q_m^*)$$

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Alice completes the retrieval by invoking the reconstruction algorithm of the emulated protocol $\mathcal{P}(\mathcal{Q}, \mathcal{A}, \mathcal{C})$ as follows:

$$\mathcal{C}(k, n/s; i, a'_1, \ldots, a'_k) = \mathcal{C}(k, n/s; i, \mathcal{A}(k, 1; x_\ell, q_1), \ldots, \mathcal{A}(k, k; x_\ell, q_k)) = x_{\ell,i}$$
What about communication cost?

In order to reduce storage overhead, we emulate a conventional PIR protocol $\mathcal{P}$ by a coded PIR protocol $\mathcal{P}^*$. How much do we pay in communication complexity?

$$U(\mathcal{P};n) \overset{\text{def}}{=} \text{Worst-case total number of bits uploaded by a protocol } \mathcal{P} \text{ for a database of length } n$$

$$D(\mathcal{P};n) \overset{\text{def}}{=} \text{Worst-case total number of bits downloaded by a protocol } \mathcal{P} \text{ for a database of length } n$$

**Theorem 4: Communication complexity of coded PIR**

Suppose there exists a $k$-server PIR code $C$ of length $m$ and dimension $s$. Then any linear $k$-server PIR protocol $\mathcal{P}$ can be emulated by a coded PIR protocol $\mathcal{P}^*$ with $s$ parts and $m$ shares, having communication complexity:

$$U(\mathcal{P}^*;n) \leq \frac{m}{k} U(\mathcal{P};n/s) + m \log k \quad \text{and} \quad D(\mathcal{P}^*;n) \leq \frac{m}{k} D(\mathcal{P};n/s)$$

**Proof.** On the upload side, the number of queries increases from $k$ to $m$, but each query is shorter as it is generated by $Q(k, n/s; i)$ rather than $Q(k, n; i)$. On the download side, the number of answers also increases from $k$ to $m$. □
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Summary of our results so far

We have shown that:

\[ \text{existing } k\text{-server linear PIR protocol } \mathcal{P} \]
\[ + \]
\[ k\text{-server PIR code } \mathcal{C} \text{ of length } m \text{ and dimension } s \]
\[ = \]
\[ \text{coded } k\text{-server PIR protocol } \mathcal{P}^* \text{ with storage overhead } m/s \]
Summary of our results so far

We have shown that:

- existing $k$-server linear PIR protocol $P$
- $k$-server PIR code $C$ of length $m$ and dimension $s$
- coded $k$-server PIR protocol $P^*$ with storage overhead $m/s$

NICE JOB ... BUT

WHAT ABOUT MY QUESTIONS!
Summary of our results so far

Why does the bit retrieval in the example work? Why does everything cancel out?

For which $m$, $s$, and $k$ do coded $k$-server PIR schemes with $s$ parts and $m$ shares exist?

What about the communication complexity of coded PIR schemes?

How small can we make the storage overhead ratio $m/s$?

We have shown that:

$$\text{existing } k\text{-server linear PIR protocol } P + k\text{-server PIR code } C \text{ of length } m \text{ and dimension } s = \text{coded } k\text{-server PIR protocol } P^* \text{ with storage overhead } m/s$$
Summary of our results so far

- Why does the bit retrieval in the example work? Why does everything cancel out?
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Summary of our results so far

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existing $k$-server linear PIR protocol $\mathcal{P}$

+ $k$-server PIR code $\mathcal{C}$ of length $m$ and dimension $s$

= coded $k$-server PIR protocol $\mathcal{P}^*$ with storage overhead $m/s$
New problem: High-rate PIR codes

According to our construction, coded PIR schemes exist whenever PIR codes exist. The storage overhead of such coded PIR schemes is completely determined by the rate of the underlying PIR code.

Open Problem: Given positive integers $s$ and $k$, determine the smallest $m$ such that there exists a $k$-server PIR code of length $m$ and dimension $s$.

$M(s,k) \overset{\text{def}}{=} \text{Shortest possible length } m \text{ of a } k\text{-server PIR code of dimension } s$

$\rho(s,k) \overset{\text{def}}{=} \text{Smallest possible redundancy of a } k\text{-server PIR code of dimension } s$

With this notation:

storage overhead $= \frac{M(s,k)}{s} = 1 + \frac{\rho(s,k)}{s}$
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With this notation:

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We have converted a PIR problem to a coding theory problem!
Optimal solution for two servers

For \( k = 2 \), the coding-theory problem is trivial. The single parity-check code of dimension \( s \) is a 2-server PIR code, and therefore:

\[
M(s, 2) = s + 1 \quad \text{and} \quad \rho(s, 2) = 1
\]

Why is this true? The encoding of each message \( x = (x_1, x_2, \ldots, x_s) \) consists of appending an overall parity bit

\[
c = x_1 + x_2 + \cdots + x_s
\]

Thus for all \( i \in [s] \), we have \( x_i = x_1 + \cdots + x_{i-1} + c + x_{i+1} + \cdots + x_s \). This corresponds to two disjoint recovery sets \( \mathcal{R}_1 = \{i\} \) and \( \mathcal{R}_2 = [s+1] \setminus \{i\} \).
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Thus for all $i \in [s]$, we have $x_i = x_1 + \cdots + x_{i-1} + c + x_{i+1} + \cdots + x_s$. This corresponds to two disjoint recovery sets $\mathcal{R}_1 = \{i\}$ and $\mathcal{R}_2 = [s+1] \setminus \{i\}$.

Theorem 5: PIR without storage overhead

For all $\varepsilon > 0$ it is possible to achieve information-theoretic PIR with communication complexity $n^{o(1)}$ by storing at most $(1 + \varepsilon)n$ bits.

**Proof.** Take $s = 1/\varepsilon$, and combine our results for $k = 2$ with the results of Dvir-Gopi on 2-server PIR with subpolynomial communication.
Open Problem: Can we achieve information-theoretic PIR with low communication cost without doubling the number of bits we need to store?
PIR codes for multiple servers

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Any reason to go on... Why not stop here?
PIR codes for multiple servers

Open Problem: Can we achieve information-theoretic PIR with low communication cost without doubling the number of bits we need to store?

Any reason to go on... Why not stop here?

As the number of servers $k$ grows, the communication complexity improves dramatically:

$$n^{O\left(\sqrt{\frac{\log \log n}{\log n}}\right)} \xrightarrow{k \text{ large}} \text{polylog}(n)$$
PIR codes for multiple servers

Open Problem: Can we achieve information-theoretic PIR with low communication cost *without doubling the number of bits we need to store*?

Any reason to go on... Why not stop here?

- As the number of servers $k$ grows, the communication complexity improves dramatically:
  $$n O\left(\sqrt{\frac{\log \log n}{\log n}}\right) \xrightarrow{k \text{ large}} \text{polylog}(n)$$

- The coding-theory problem of determining $M(s,k)$ becomes much more interesting for $k \geq 3$. It has strong connections with:
  - Steiner systems and $t$-designs
  - majority-logic decodable codes
  - local codes with availability
  - multiset batch codes
  - bipartite graphs of girth 6
  - constant-weight codes
PIR codes for multiple servers

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  - Steiner systems and $t$-designs
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  - constant-weight codes
Suppose that \( k = 3 \) and \( s = \sigma^2 \) for some \( \sigma \in \mathbb{Z} \). Arrange the \( \sigma^2 \) message bits in the form of a \( \sigma \times \sigma \) square. To every message, we append \( 2\sigma \) parity bits given by:

\[
\begin{align*}
  c_i &= x_{i,1} + x_{i,2} + \cdots + x_{i,\sigma} \quad \text{for } i \in [\sigma] \\
  c'_j &= x_{1,j} + x_{2,j} + \cdots + x_{\sigma,j} \quad \text{for } j \in [\sigma]
\end{align*}
\]

More generally, we arrange \( \sigma^{k-1} \) message bits in the form of a \((k-1)\)-dimensional hypercube and append a parity bit to each of its \((k-1)\) columns. This proves:

\[
M(s, k) = s + (k-1) \left\lceil \sqrt[k-1]{s} \right\rceil \leq (k-1) \left\lceil \sqrt[k-1]{s} \right\rceil
\]

It follows that \( \lim_{s \to \infty} \frac{M(s, k)}{s} = 1 \) for all fixed \( k \geq 2 \). Therefore, we have proved:

**Corollary 6:** Multiple-server PIR without storage overhead

For all fixed \( k \geq 2 \) and all \( \varepsilon > 0 \), there exist \( k \)-server coded PIR schemes that store at most \( (1 + \varepsilon)n \) bits.
The hypercube construction

Suppose that \( k = 3 \) and \( s = \sigma^2 \) for some \( \sigma \in \mathbb{Z} \). Arrange the \( \sigma^2 \) message bits in the form of a \( \sigma \times \sigma \) square. To every message, we append \( 2\sigma \) parity bits given by:

\[
\begin{align*}
c_i &= x_{i,1} + x_{i,2} + \cdots + x_{i,\sigma} \quad \text{for } i \in [\sigma] \\
c'_j &= x_{1,j} + x_{2,j} + \cdots + x_{\sigma,j} \quad \text{for } j \in [\sigma]
\end{align*}
\]

Then for each message bit \( x_{i,j} \) we have three disjoint recovery equations given by \( x_{i,j} \) itself and:

\[
\begin{align*}
x_{i,1} + \cdots + x_{i,j-1} + c_i + x_{i,j+1} + \cdots + x_{i,\sigma} &= x_{1,j} + \cdots + x_{i-1,j} + c'_j + x_{i+1,j} + \cdots + x_{\sigma,j}
\end{align*}
\]
Suppose that $k = 3$ and $s = \sigma^2$ for some $\sigma \in \mathbb{Z}$. Arrange the $\sigma^2$ message bits in the form of a $\sigma \times \sigma$ square. To every message, we append $2\sigma$ parity bits given by:

$$c_i = x_{i,1} + x_{i,2} + \cdots + x_{i,\sigma} \quad \text{for } i \in [\sigma]$$

$$c'_j = x_{1,j} + x_{2,j} + \cdots + x_{\sigma,j} \quad \text{for } j \in [\sigma]$$

Then for each message bit $x_{i,j}$ we have three disjoint recovery equations given by $x_{i,j}$ itself and:

$$x_{i,1} + \cdots + x_{i,j-1} + c_i + x_{i,j+1} + \cdots + x_{i,\sigma} = x_{1,j} + \cdots + x_{i-1,j} + c'_j + x_{i+1,j} + \cdots + x_{\sigma,j}$$

More generally, we arrange $\sigma^{k-1}$ message bits in the form of a $(k-1)$-dimensional hypercube and append a parity bit to each of its $(k-1)\sigma^{k-2}$ columns. This proves:

$$M(s, k) = s + (k-1)\left\lceil \sqrt[k-1]{s} \right\rceil^{k-2}$$

$$\rho(s, k) \leq (k-1)\left\lceil \sqrt[k-1]{s} \right\rceil^{k-2}$$
The hypercube construction

Suppose that \( k = 3 \) and \( s = \sigma^2 \) for some \( \sigma \in \mathbb{Z} \). Arrange the \( \sigma^2 \) message bits in the form of a \( \sigma \times \sigma \) square. To every message, we append \( 2\sigma \) parity bits given by:

\[
\begin{align*}
    c_i &= x_{i,1} + x_{i,2} + \cdots + x_{i,\sigma} \quad \text{for } i \in [\sigma] \\
    c'_j &= x_{1,j} + x_{2,j} + \cdots + x_{\sigma,j} \quad \text{for } j \in [\sigma]
\end{align*}
\]

Then for each message bit \( x_{i,j} \) we have three disjoint recovery equations given by \( x_{i,j} \) itself and:

\[
x_{i,1} + \cdots + x_{i,j-1} + c_i + x_{i,j+1} + \cdots + x_{i,\sigma} = x_{1,j} + \cdots + x_{i-1,j} + c'_j + x_{i+1,j} + \cdots + x_{\sigma,j}
\]

More generally, we arrange \( \sigma^{k-1} \) message bits in the form of a \((k-1)\)-dimensional hypercube and append a parity bit to each of its \((k-1)\sigma^{k-2}\) columns. This proves:

\[
\begin{align*}
    M(s, k) &= s + (k-1) \left\lceil \frac{k-1}{\sqrt{s}} \right\rceil^{k-2} \\
    \rho(s, k) &\leq (k-1) \left\lceil \frac{k-1}{\sqrt{s}} \right\rceil^{k-2}
\end{align*}
\]

It follows that \( \lim_{s \to \infty} M(s, k) / s = 1 \) for all fixed \( k \geq 2 \). Therefore, we have proved:

**Corollary 6: Multiple-server PIR without storage overhead**

For all fixed \( k \geq 2 \) and all \( \varepsilon > 0 \), there exist \( k \)-server coded PIR schemes that store at most \((1 + \varepsilon)n\) bits.
 Majority-logic decodable codes

Majority-logic decoding originated with the work of Reed and Massey over 50 years ago. 100s of papers in the 1960s and 1970s... now forgotten.
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**Definition: Majority-logic decodable codes**

A linear code $C$ of length $n$ is **majority-logic decodable** with parameter $J$ iff for each position $i \in [n]$, there exist $J$ parity-checks orthogonal on this position:

$$
\begin{align*}
    i & \\
    11\ldots1 & \\
    111\ldots1 & \\
    \vdots & \\
    1 & 11\ldots1 \\
    1 & 11\ldots1
\end{align*}
$$

$J$ codewords of $C^\perp$
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$$\begin{align*}
\text{11...11} \\
\text{111...1} \\
\vdots \\
\text{11...1} \\
\text{11...1} \\
\end{align*}$$

$J$ codewords of $C^\perp$

**Majority-logic decoding:** Given $y$, evaluate the $J$ orthogonal parity-checks for each position $i$. 

Diagram: A coded vector $c$ is corrupted by $t$ errors, resulting in a received vector $y$. The majority-logic decoder determines the correct codeword $y_i$ for each position $i$. 

If $y_i = c_i$ and $\leq t$ other errors, then at least $J-t$ checks evaluate to 0. If $y_i \neq c_i$ and $\leq t$ other errors, then at least $J-t$ checks evaluate to 1.

There is an error at position $i$ iff a majority of the $J$ checks evaluate to 1.
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\[
\begin{pmatrix}
  i \\
  11\ldots1 \\
  111\ldots1 \\
  \vdots \\
  1 \\
  1 \\
\end{pmatrix}
\]

\[
\begin{pmatrix}
  11\ldots1 \\
  111\ldots1 \\
  \vdots \\
  11\ldots1 \\
  11\ldots1 \\
\end{pmatrix}
\]

\[
\text{J codewords of } C^\perp
\]

**Majority-logic decoding:** Given $y$, evaluate the $J$ orthogonal parity-checks for each position $i$. Then:

- $y_i = c_i$ and $\leq t$ other errors $\implies$ at least $J - t$ checks evaluate to 0
Majority-logic decodable codes

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$$\begin{align*}
  i \\
  11\ldots1 \\
  11\ldots1 \\
  \vdots \\
  1 \\
  1 \\
\end{align*}$$

$J$ codewords of $C^\perp$

**Majority-logic decoding:** Given $y$, evaluate the $J$ orthogonal parity-checks for each position $i$. Then:

- $y_i = c_i$ and $\leq t$ other errors $\implies$ at least $J - t$ checks evaluate to 0
- $y_i \neq c_i$ and $\leq t$ other errors $\implies$ at least $J - t$ checks evaluate to 1
Majority-logic decodable codes

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Definition: Majority-logic decodable codes

A linear code $C$ of length $n$ is \textbf{majority-logic decodable} with parameter $J$ iff for each position $i \in [n]$, there exist $J$ parity-checks orthogonal on this position:

\[
\begin{array}{r}
\text{i} \\
11...11 \\
111...1 \\
\vdots \\
1 \\
1
\end{array} \quad \begin{array}{r}
\text{J codewords of } C^\perp
\end{array}
\]

\[\text{errors} \quad \rightarrow \quad y\]

\[c \quad \rightarrow \quad y\]

\[c_i \quad ? \quad y_i\]

\textbf{Majority-logic decoding}: Given $y$, evaluate the $J$ orthogonal parity-checks for each position $i$. Then:

- $y_i = c_i$ and $\leq t$ other errors $\implies$ at least $J - t$ checks evaluate to 0
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There is an error at position $i$ iff a \textit{majority of the J checks} evaluate to 1.
Lemma 7: PIR codes from majority-logic codes

Let $C$ be a majority-logic decodable code with parameter $J$. Then $C$ is also a $k$-server PIR code with $k = J + 1$.

**Proof.** It is easy to see that a systematic generator matrix $G$ for $C$ has property $P_k$ with $k = J + 1$. Since $G$ is systematic, the column in position $i$ is $e_i$. 

$$G = \begin{bmatrix}
11\ldots1 & 1 \\
1 & 11\ldots1 \\
\vdots & \ddots \\
1 & 11\ldots1 \\
1 & 1 \\
\end{bmatrix}$$

The $J$ codewords of $C^\perp$
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$$

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\[
G = \begin{bmatrix}
R_1 & e_i & R_2 & \cdots & R_{J-1} & R_J \\
11\ldots1 & 1 & 11\ldots1 & \cdots & 11\ldots1 & 11\ldots1
\end{bmatrix}
\]

Thus $\{e_i\}$ and $R_1, R_2, \ldots, R_J$ are **disjoint sets of columns** of $G$ that add to $e_i$. \(\square\)
**Lemma 7: PIR codes from majority-logic codes**

Let $C$ be a majority-logic decodable code with parameter $J$. Then $C$ is also a $k$-server PIR code with $k = J + 1$.

Numerous algebraic constructions of cyclic majority-logic decodable codes are known. For example, Reed-Muller codes, BCH codes, and other codes invariant under the group of affine permutations:

$$\alpha^i \mapsto \beta \alpha^i + \gamma \quad \text{for all} \quad i = 0, 1, \ldots, 2^m - 2 \quad \text{and} \quad \beta, \gamma \in \text{GF}(2^m)$$

T. Kasami, S. Lin, and W.W. Peterson, Some results on cyclic codes which are invariant under the affine group, *Information and Control*, vol. 2, pp. 475–496, November 1968.

**Theorem: Doubly-transitive majority-logic codes**

Let $n = 2^{2ab} - 1$ and let $C$ be a binary cyclic code of length $n$ and co-dimension $(2^{b+1} - 1)^a - 1$. Then $C$ is majority-logic decodable with parameter $J = 2^a + 1$. 
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As a corollary to this theorem and Lemma 7, whenever the number of servers is of the form $k = 4, 6, 10, \ldots, 2^a + 2$, we have:

$$\rho(s, k) = O(\sqrt{s})$$
**Construction from certain set systems**

**Definition: Almost disjoint $k$-covers**

Let $\mathcal{A} = \{A_1, A_2, \ldots, A_r\}$ be a collection of subsets of $[s]$. We say that $\mathcal{A}$ is a *k-cover of $[s]$* if every $i \in [s]$ belongs to at least $k$ of the subsets in $\mathcal{A}$. We say that these subsets are **almost disjoint** if any two of them intersect in at most one element.

Given any collection $\mathcal{A} = \{A_1, A_2, \ldots, A_r\}$ of subsets of $[s]$, we construct a systematic $(s + r, s)$ linear code $C(\mathcal{A})$ as follows. To each message $x = (x_1, x_2, \ldots, x_s)$, we append $r$ parity bits given by:

$c_1 = \sum_{j \in A_1} x_j, \quad c_2 = \sum_{j \in A_2} x_j, \quad \cdots, \quad c_r = \sum_{j \in A_r} x_j$
Construction from certain set systems

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$$
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$$

**Lemma 8: PIR codes from almost disjoint $k$-covers**

Suppose that $\mathcal{A} = \{A_1, A_2, \ldots, A_r\}$ is a $(k-1)$-cover of $[s]$ and the sets in $\mathcal{A}$ are almost disjoint. Then the resulting $(s + r, s)$ code $C(\mathcal{A})$ is a $k$-server PIR code.
Construction from certain set systems

**Definition: Almost disjoint \(k\)-covers**

Let \(A = \{A_1, A_2, \ldots, A_r\}\) be a collection of subsets of \([s]\). We say that \(A\) is a \(k\)-cover of \([s]\) if every \(i \in [s]\) belongs to at least \(k\) of the subsets in \(A\). We say that these subsets are **almost disjoint** if any two of them intersect in at most one element.

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\[
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\]

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Suppose that \(A = \{A_1, A_2, \ldots, A_r\}\) is a \((k-1)\)-cover of \([s]\) and the sets in \(A\) are almost disjoint. Then the resulting \((s + r, s)\) code \(C(A)\) **is a \(k\)-server PIR code**.

**Proof.** Given \(i \in [s]\), find \(k - 1\) subsets in \(A\) that contain \(i\). W.l.o.g., suppose these subsets are \(A_1, A_2, \ldots, A_{k-1}\). Let \(A'_j = A_j \setminus \{i\}\) for all \(j\). Then the sets \(A'_1, A'_2, \ldots, A'_{k-1}\) are disjoint. These sets give rise to \(k\) disjoint recovery equations:

\[
    x_i = c_1 + \sum_{j \in A'_1} x_j = c_2 + \sum_{j \in A'_2} x_j = \cdots = c_{k-1} + \sum_{j \in A'_{k-1}} x_j
\]
Construction from certain set systems

Definition: Almost disjoint $k$-covers

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\[
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\]

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Corollary 9: PIR codes from almost disjoint $k$-covers

If there exists an almost disjoint $(k-1)$-cover of $[s]$ with $r$ sets, then $\rho(s, k) \leq r$. 

Construction from certain set systems

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If there exists an almost disjoint $(k-1)$-cover of $[s]$ with $r$ sets, then $\rho(s, k) \leq r$.

Where can we get almost disjoint $k$-covers or small size $r$?
Let $V$ be a set with $r$ elements, called **points**. A **Steiner system** $S(2, q, r)$ is a collection $\mathcal{B}$ of subsets of $V$ of size $q$, called **blocks**, such that every pair of points is contained in exactly one block.

Such a system is an example of a **balanced incomplete block design**.
Let $V$ be a set with $r$ elements, called points. A Steiner system $S(2, q, r)$ is a collection $\mathcal{B}$ of subsets of $V$ of size $q$, called blocks, such that every pair of points is contained in exactly one block.

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**Observation:** There are $b = \binom{r}{2}/\binom{q}{2}$ blocks in $B$ and each point is contained in $(r-1)/(q-1)$ of them. Moreover, any two blocks intersect in at most one point.
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**Conclusion:** The blocks of a Steiner system $S(2, q, s)$ form an almost disjoint $(s-1)/(q-1)$-cover of $[s]$. Therefore, when such Steiner systems exist, we have

$$\rho(s, k) \leq \text{number of blocks in } S(2, q, s) = \frac{s(s-1)}{q(q-1)} = \frac{s(k-1)^2}{s+k}$$

where $k = (s-1)/(q-1) + 1$. 
PIR codes from Steiner systems

Let $V$ be a set with $r$ elements, called points. A Steiner system $S(2, q, r)$ is a collection $B$ of subsets of $V$ of size $q$, called blocks, such that every pair of points is contained in exactly one block. Such a system is an example of a balanced incomplete block design.

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where $k = (s-1)/(q-1) + 1$. But, by Fisher’s inequality ($\# \text{blocks} \geq \# \text{points}$), this gives $\rho(s, k) \leq s$ at best.

💡 We can do much better with Steiner systems!
PIR codes from Steiner systems

Let $V$ be a set with $r$ elements, called points. A **Steiner system** $S(2,q,r)$ is a collection $B$ of subsets of $V$ of size $q$, called blocks, such that every pair of points is contained in exactly one block.

- Such a system is an example of a *balanced incomplete block design*.

**Example: Fano plane $S(2, 3, 7)$**

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By Wilson's theorem, a Steiner system $S(2,q,r)$ exists for all sufficiently large $r$ whenever $(q-1) | (r-1)$ and $q(q-1) | r(r-1)$. Combining this theorem with Lemma 10 and Corollary 9, we have:

$$\rho(s,k) = O(\sqrt{s})$$

for all fixed $k$. 

PIR codes from Steiner systems

Let $V$ be a set with $r$ elements, called **points**. A **Steiner system** $S(2, q, r)$ is a collection $\mathcal{B}$ of subsets of $V$ of size $q$, called **blocks**, such that every pair of points is contained in exactly one block.

- Such a system is an example of a *balanced incomplete block design*.

**Example: Fano plane $S(2, 3, 7)$**

![Fano plane diagram]

**Lemma 10: PIR codes from Steiner systems**

Let $S(2, q, r)$ be a Steiner system. For each $v \in V$, let $A_v \subset \mathcal{B}$ be the set of blocks that contain $v$. Then the sets $\{A_v : v \in V\}$ form an almost disjoint $q$-cover of $[b]$.

**Proof.** For any pair of points $u$ and $v$, there is only one block that contains both. Hence $|A_v \cap A_u| = 1$, and the sets $\{A_v : v \in V\}$ are almost disjoint. □

By Wilson’s theorem, a Steiner system $S(2, q, r)$ exists for all sufficiently large $r$ whenever $(q-1)|(r-1)$ and $q(q-1)|r(r-1)$. Combining this theorem with Lemma 10 and Corollary 9, we have:

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**Example: Fano plane $S(2, 3, 7)$**

![Diagram of the Fano plane](image)

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By Wilson’s theorem, a Steiner system $S(2, q, r)$ exists for all sufficiently large $r$ whenever $(q-1)|(r-1)$ and $q(q-1)|r(r-1)$. Combining this theorem with Lemma 10 and Corollary 9, we have:

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for all fixed $k$. 


PIR codes from bipartite graphs

Let $G = (U,V; E)$ be a bipartite graph, with bipartition $U, V$ and edge set $E$. We consider the neighborhoods $N(v) = \{u \in U : (u,v) \in E\}$ of vertices in $V$.

**Lemma 11: PIR codes from bipartite graphs**

*If $G$ has no 4-cycles, then the neighborhoods of vertices in $V$, namely the set $\{N(v) : v \in V\}$, form an almost disjoint $k$-cover of $U$, where $k = \min_{u \in U} \deg(u)$.*

**Proof.** Assume to the contrary that there are vertices $v_1, v_2 \in V$ such that $|N(v_1) \cap N(v_2)| \geq 2$. Let $u_1, u_2$ be some two vertices in $N(v_1) \cap N(v_2)$. Then the induced subgraph on $\{v_1, v_2, u_1, u_2\}$ is $K_{2,2}$ which is a 4-cycle in $G$. □

**Note:** [DGRS15] use a similar construction for batch codes, but with $\text{girth}(G) \geq 8$. 
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<table>
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<tr>
<th>W</th>
<th>9x252</th>
<th>PIR codes from bipartite graphs</th>
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<tbody>
<tr>
<td>W</td>
<td>10x229</td>
<td>Let $G = (U, V; E)$ be a bipartite graph, with bipartition $U, V$ and edge set $E$. We consider the neighborhoods $N(v) = { u \in U : (u, v) \in E }$ of vertices in $V$.</td>
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<td>W</td>
<td>10x198</td>
<td><strong>Lemma 11: PIR codes from bipartite graphs</strong></td>
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<tr>
<td>W</td>
<td>10x182</td>
<td>If $G$ has no 4-cycles, then the neighborhoods of vertices in $V$, namely the set ${N(v) : v \in V}$, form an almost disjoint $k$-cover of $U$, where $k = \min_{u \in U} \deg(u)$.</td>
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<td>W</td>
<td>11x138</td>
<td><strong>Proof.</strong> Assume to the contrary that there are vertices $v_1, v_2 \in V$ such that $</td>
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<td>W</td>
<td>42x9</td>
<td><strong>Note:</strong> [DGRS15] use a similar construction for batch codes, but with $\text{girth}(G) \geq 8$.</td>
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**Lemma 11: PIR codes from bipartite graphs**

If $G$ has **no 4-cycles**, then the neighborhoods of vertices in $V$, namely the set $\{N(v) : v \in V\}$, form an almost disjoint $k$-cover of $U$, where $k = \min_{u \in U} \deg(u)$.

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Given $s$ and $k$, we would like to construct a bipartite graph $G = (U,V; E)$ with the following properties:

$$|U| = s \quad \min_{u \in U} \deg(u) = k - 1 \quad \text{girth}(G) \geq 6$$

If we can do this, then $\rho(s,k) \leq |V|$ by Corollary 10. What is the least possible number of vertices in $V$ for such a graph?

**Note:** [DGRS15] use a similar construction for batch codes, but with $\text{girth}(G) \geq 8$. 

---

**PIR codes from bipartite graphs**
Let $G = (U, V; E)$ be a bipartite graph, with bipartition $U, V$ and edge set $E$. We consider the neighborhoods $N(v) = \{ u \in U : (u, v) \in E \}$ of vertices in $V$.

**Lemma 11:** PIR codes from bipartite graphs

*If $G$ has no 4-cycles, then the neighborhoods of vertices in $V$, namely the set $\{ N(v) : v \in V \}$, form an almost disjoint $k$-cover of $U$, where $k = \min_{u \in U} \deg(u)$.***

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If we can do this, then $\rho(s, k) \leq |V|$ by Corollary 10. What is the least possible number of vertices in $V$ for such a graph? Using the best known results on bipartite cages, we get:

$$\rho(s, k) = O(\sqrt{s}) \quad \text{for all fixed } k$$

**Note:** [DGRS15] use a similar construction for batch codes, but with $\text{girth}(G) \geq 8.$
Definition: Constant-weight codes

Let $A_2(n, d, w)$ be the number of codewords in the largest binary code $C$ of length $n$ and minimum distance $d$ such that all the codewords of $C$ have weight $w$. 
PIR codes from constant-weight codes

Definition: Constant-weight codes

Let $A_2(n, d, w)$ be the number of codewords in the largest binary code $C$ of length $n$ and minimum distance $d$ such that all the codewords of $C$ have weight $w$. 

For example, for $k = 3$ we conclude that $\rho(s, 3)$ is upper bounded by the smallest $n$ such that $n(n-1) \geq 2s$. In general, we again have $\rho(s, k) = O(\sqrt{s})$. 
Definition: Constant-weight codes

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Observations:

- $d = 2w$ if and only if any two codewords have disjoint supports:

![Disjoint Supports Diagram]

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- $d = 2w - 2$ iff any two codewords intersect in at most one position:
PIR codes from constant-weight codes

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Observations:
- $d = 2w$ if and only if any two codewords have disjoint supports:
- $d = 2w - 2$ iff any two codewords intersect in at most one position:

Now let $s = A_2(n, 2w-2, w)$, and consider the $s \times n$ matrix having the codewords of $C$ as its rows:

As the weight of each row is $w$, columns form a $w$-cover of $[s]$. 
PIR codes from constant-weight codes

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Let $A_2(n, d, w)$ be the number of codewords in the largest binary code $C$ of length $n$ and minimum distance $d$ such that all the codewords of $C$ have weight $w$.

**Observations:**

- $d = 2w$ if and only if any two codewords have disjoint supports:

  ![Disjoint Supports](image)

- $d = 2w - 2$ iff any two codewords intersect in at most one position:

  ![Intersection in at most one position](image)

Now let $s = A_2(n, 2w-2, w)$, and consider the $s \times n$ matrix having the codewords of $C$ as its rows:

![Matrix](image)

As the row supports are almost disjoint, the column supports are also almost disjoint.
PIR codes from constant-weight codes

Definition: Constant-weight codes

Let \( A_2(n, d, w) \) be the number of codewords in the largest binary code \( C \) of length \( n \) and minimum distance \( d \) such that all the codewords of \( C \) have weight \( w \).

Now let \( s = A_2(n, 2w-2, w) \), and consider the \( s \times n \) matrix having the codewords of \( C \) as its rows:

As the row supports are almost disjoint, the column supports are also almost disjoint.

Theorem 12: PIR codes from constant-weight codes

\[ \rho(s, k) \leq \text{the smallest } n \text{ such that } A_2(n, 2k-4, k-1) \geq s \]

Supports are almost disjoint \( \uparrow \)

\( (k-1) \)-cover of \([s]\)
**Definition: Constant-weight codes**

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As the row supports are almost disjoint, the column supports are also almost disjoint.

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For example, for $k = 3$ we conclude that $\rho(s, 3)$ is upper bounded by the smallest $n$ such that $n(n-1) \geq 2s$. In general, we again have $\rho(s, k) = O(\sqrt{s})$. 
### Tables of short PIR codes

**number\(k\) of servers emulated**

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**Redundancy** \(\rho(s,k)\) of the best-known PIR codes
### Tables of short PIR codes

#### number $k$ of servers emulated

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#### Redundancy $\rho(s,k)$ of the best-known PIR codes

*Improve any entry in this table!*
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### Storage overhead of the best-known PIR codes
Thank you for your attention!

Please send you queries...